A note on the number of additive triples in subsets of integers

Katherine Staden*
Mathematics Institute and DIMAP
University of Warwick
Coventry CV4 7AL, UK

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Abstract

An additive triple in a set A of integers is a subset $\{x, y, z\} \subseteq A$ such that $x + y = z$. In this note, we precisely count the both the minimum and maximum number of additive triples among all subsets of the first n positive integers of a given size. We further determine the extremal subsets.

1 Introduction

An *additive triple*, or *Schur triple*, or *sum* in a set A of integers is a subset $\{x, y, z\} \subseteq A$ such that $x+y = z$. Note that x, y, z need not be distinct. Given $n \geq m \geq 1$, write $[m, n] := \{m, \dots, n\}$ and $[n] := [1, n]$. The size and structure of the largest sum-free subsets of $[n]$ are well known. Indeed, every such subset has size $\lceil n/2 \rceil$ and the unique extremal subsets are $O = \{1, 3, 5, \dots, 2\lceil n/2 \rceil - 1\}$ and $[\lfloor n/2 \rfloor + 1, n]$; and additionally $[n/2, n - 1]$ if n is even.

In this note we are interested in the more general question of counting the number of sums in a set of a fixed size $a \in [n]$. Given $A \subseteq [n]$, define

$$s(A) := \left|\left\{(x, y, z) \in A^3 : x + y = z \text{ and } x \leq y\right\}\right|$$

to be the number of additive triples in A. Let $s^-(a)$ be the minimum of $s(A)$ and $s^+(a)$ the maximum over all subsets $A \subseteq [n]$ of size a. So $s^-(a) > 0$ if and only if $a > \lceil n/2 \rceil$. We prove that in fact, when $a > \lceil n/2 \rceil$, we have $s^-(a) = (a-\lceil n/2 \rceil)(a-\lfloor n/2 \rfloor)$, and, moreover, there is a unique minimising set (i.e. a set which attains this bound). The situation for $s^+(a)$ is only slightly more complicated, and there may be multiple maximising sets, but they have a very similar structure. Given a set X of natural numbers, write $b \cdot X := \{bx : x \in X\}$. The purpose of this note is to prove the following result.

Theorem 1 *For all positive integers $n \geq a$,*

(i) $s^-(a) = \max\{0, (a - \lfloor n/2 \rfloor)(a - \lceil n/2 \rceil)\}$, and if $a > \lceil n/2 \rceil$ then the unique minimising set is $[n - a + 1, n]$.

*Email: `k.l.staden@warwick.ac.uk`. Supported by ERC grant 306493.

(ii) $s^+(a) = \lfloor a^2/4 \rfloor$ *and the only maximising sets are of the form* $\ell \cdot [a]$*, or if* a *is odd additionally* $\ell \cdot ([a-1] \cup \{a+1\})$*, for* $\ell \in \mathbb{N}$*.*

Notice that, if $a \leq \lceil n/2 \rceil$, there is more than one (usually many) sum-free subsets of $[n]$, and we do not provide a characterisation for minimising sets here. The conclusions of Theorem 1 are not particularly surprising. One might expect that, in order to obtain a minimising set of size $a > \lceil n/2 \rceil$, one should add appropriate elements to a sum-free set of largest size $\lceil n/2 \rceil$. This is indeed the case, and the set is $[\lfloor n/2 \rfloor + 1, n]$; but note that adding a single new element to the other sum-free set of largest size, the set of odd integers, gives rise to many additive triples. For the maximisation problem, $[a]$ is a natural candidate, and it is clear that dilations have no effect. Note that here n is redundant; its only effect is to upper bound ℓ.

The questions addressed in this paper are instances of the so-called *supersaturation* problem which has been studied in several contexts in discrete mathematics. It was initiated by Erdős in 1955 [2] who asked for the minimum number of triangles one can guarantee in a graph of given order and size. This problem was solved asymptotically by Razborov [4] only in 2008. In contrast, the maximisation problem for triangles is relatively simple. See Chapter VII in [1] for more details on the history of and recent progress on this fascinating problem in graph theory.

Samotij and Sudakov [6] initiated the study of exact supersaturation in finite abelian groups. That is, given an (additive) abelian group Γ, and $a \in [|\Gamma|]$, they asked for the minimum number of (ordered) additive triples in subsets A of Γ of size a. They were able to solve the problem completely in the case when $\Gamma = \mathbb{Z}_p$ for prime p, and in the hypercube $\mathbb{Z}_2^n$ for all natural numbers n. They obtained other partial results and explained why this problem seems rather hard to solve for general abelian groups Γ. Theorem 1 shows that, in the integer setting, the situation is very much simpler.

A question related to determining $s^-(a)$ was posed by Graham, Rödl and Ruciński in [3]. They asked for the minimum number of monochromatic additive triples among all 2-colourings of $[n]$. This was solved independently by Schoen [7] and by Robertson and Zeilberger [5] who obtained the tight lower bound of $n^2/22 + O(n)$. Schoen was further able to describe the structure of extremal colourings, and these bear a strong resemblance to the extremal sets in Theorem 1: one colour class is essentially the union of an a_1-sized minimising set and an a_2-sized maximising set, for optimised a_1, a_2 (about $n/11$ and $4n/11$ respectively).

2 The proof of Theorem 1

The proof of Theorem 1 is elementary. Let us briefly describe the main idea for (i) (the proof of (ii) is very similar). We choose an $a \in [n]$ and corresponding hypothetical 'worst counterexample' A of size a. This means that, when compared to the conjectured minimising subset $[n-a+1, n]$ of the same size, the set A contains the fewest number of additive triples. Then every element of A must lie in few additive triples, or we could remove it and find a worse counterexample. Then it is simply a matter of considering an element $y \in A$ which should lie in many additive triples (the smallest element of A is a natural choice), and showing that it must in fact lie in too many, or $A \setminus \{y\}$ is also a worst counterexample.

First we do some easy calculations to determine $s([n-a+1, n])$ and $s([a])$.

Proposition 2 *For all positive integers $n \geq a$ we have*

$$s([n-a+1,n]) = \left(a - \left\lfloor \frac{n}{2} \right\rfloor\right)\left(a - \left\lceil \frac{n}{2} \right\rceil\right),$$
$$s([a]) = \left\lfloor \frac{a^2}{4} \right\rfloor,$$

and if a is odd, then $s([a-1] \cup \{a+1\}) = s([a])$.

Proof. For each $i \in [\lceil n/2 \rceil + 1, a]$, the number of additive triples in $[n-a+1,n]$ whose smallest element is $n-i+1$ is $|[n-i+1,i-1]| = 2i-n-1$. If $i \in [\lceil n/2 \rceil]$ then there are no such additive triples. So

$$s([n-a+1,n]) = \sum_{i=\lceil n/2 \rceil+1}^{a} (2i-n-1) = (a - \lceil n/2 \rceil)(a - \lfloor n/2 \rfloor),$$

as required. For each $i \in [\lfloor a/2 \rfloor]$, the number of additive triples in $[a]$ whose smallest element is i is $|[i,a-i]| = a+1-2i$. If $i \in [\lfloor a/2 \rfloor + 1, a]$ then there are no such additive triples. So

$$s([a]) = \sum_{i \in [\lfloor a/2 \rfloor]} (a+1-2i) = \lfloor a^2/4 \rfloor.$$

The final assertion follows from the fact that, when a is odd, both a and $a+1$ lie in the same number of additive triples whose other members lie in $[a-1]$. ∎

Given a set $A \subseteq [n]$ and $y \in A$, we write

$$s_A(y) := s(A) - s(A \setminus \{y\})$$

for the number of additive triples in A which contain y. Given $B \subseteq A \setminus \{y\}$, write

$$s_A(y,B) := s_A(y) - s_{A \setminus B}(y)$$

for the number of additive triples in A which contain y and at least one element from B.

Proof of Theorem 1. We first prove (i). Suppose that (i) is not true for some fixed $n \in \mathbb{N}$. For all $y \in [n]$, write $I_y := [n-y+1,n]$. Among all $a \in [\lceil n/2 \rceil + 1, n]$ consider only those for which (i) does not hold and the difference $s(I_a) - s^-(a) \geq 0$ is maximised. Among these, choose the smallest such a. Then there is a set $A \subseteq [n]$ of size a such that $s(A) = s^-(a)$, and moreover, either $s(A) < s(I_a)$; or $s(A) = s(I_a)$ but $A \neq I_a$.

Let $y \in A$ be arbitrary. By the choice of a, we have that

$$s(A) - s(I_a) = s^-(a) - s(I_a) \leq s^-(a-1) - s(I_{a-1}) \leq s(A \setminus \{y\}) - s(I_{a-1}), \tag{1}$$

noting that if $a = \lceil n/2 \rceil + 1$ then $s^-(a) - s(I_a) \leq 0 = s^-(a-1) - s(I_{a-1})$. Then

$$s_A(y) \leq s(I_a) - s(I_{a-1}) = s_{I_a}(n-a+1) = |[n-a+1,a-1]| = 2a-n-1. \tag{2}$$

Now let b be such that $\min(A) = n-b+1$. So $b \geq a+1$. We have that $s_{I_b}(n-b+1) = 2b-n-1$. For every $x \in I_b \setminus A$, there are at most two additive triples $\{n-b+1,x,z\}$: namely taking $z = n-b+1+x$ and $z = x-(n-b+1)$. Thus

$$s_{I_b}(n-b+1, I_b \setminus A) \leq 2(|I_b \setminus A|) = 2(b-a). \tag{3}$$

So

$$s_A(n-b+1) \geq s_{I_b}(n-b+1) - 2(b-a) = 2b-n-1-2(b-a) = 2a-n-1. \quad (4)$$

Thus we have equality in (1)–(4) for $y = n-b+1$. So (1) implies that $s(A) = s^-(a)$ and $s(A') = s^-(a-1)$ where $A' := A \setminus \{n-b+1\}$. That is, A' is a minimising set of size $a-1$. Since a is minimal, we have either $A' := I_{a-1}$; or $a = \lceil n/2 \rceil + 1$ and A' is a sum-free set of size $\lceil n/2 \rceil$.

Consider the latter case first. Then $s_A(n-b+1) = 2\lceil n/2 \rceil - n + 1$, and $A' = O$ (the set of odds); or $A' = [\lfloor n/2 \rfloor + 1, n]$; or $A' = [n/2, n-1]$ if n is even. We cannot have $A' = O$ since $n-b+1 \in [n] \setminus A'$ and $n-b+1 < \min(A') = 1$. When n is even, we cannot have $A' = [n/2, n-1]$, since then $n-b+1 \leq n/2-1$ and $s_A(n-b+1) \geq 2 > 2(n/2)-n+1$. If $A' = [\lfloor n/2 \rfloor + 1, n]$, then it is easy to see that $s_A(n-b+1) = 2\lceil n/2 \rceil - n + 1$ if and only if $n-b+1 = \lfloor n/2 \rfloor$. So $A = [\lfloor n/2 \rfloor, n] = I_{\lceil n/2 \rceil + 1}$, a contradiction.

Otherwise, $A' = I_{a-1}$. But then

$$2a-n-1 = s_{I_{a-1} \cup \{n-b+1\}}(n-b+1) = 1 + |[n-a+2, b-1]| = b-n+a-1,$$

so $b = a$, a contradiction to $A \neq I_a$. This completes the proof of (i).

We now turn to (ii), and begin by considering some small values of a. The statement is vacuous when $a = 1$. For $a = 2$, we have for $x < y$ that $s(\{x,y\}) \leq 1$ with equality if and only if $y = 2x$. Suppose now that $a = 3$, let $x < y < z$ and let $A := \{x,y,z\}$. The potential sums are $x+x=y$, $x+x=z$, $x+y=z$ and $y+y=z$. At most two of these can hold simultaneously, with equality if and only if $y = 2x$ and $z = x+y = 3x$; or $y = 2x$ and $z = 2y = 4x$. Thus $A = x \cdot [3]$ or $x \cdot \{1,2,4\}$, as required.

Suppose that (ii) is not true for some fixed $n \in \mathbb{N}$. Among all $a \in [n]$ consider only those for which (ii) does not hold and the difference $s^+(a) - s([a]) \geq 0$ is maximised. Among these, choose the smallest such a. Then there is a set $A \subseteq [n]$ of size a such that $s(A) = s^+(a)$, and moreover, either $s(A) > s([a])$; or $s(A) = s([a])$ but there is no $\ell \in [n]$ for which $A = \ell \cdot [a]$ or $A = \ell \cdot ([a-1] \cup \{a+1\})$. By the above discussion, we may assume that $a \geq 4$.

Let $y \in A$ be arbitrary. We have by the choice of a that

$$s(A) - s([a]) = s^+(a) - s([a]) \geq s^+(a-1) - s([a-1]) \geq s(A \setminus \{y\}) - s([a-1]). \quad (5)$$

So

$$s_A(y) = s(A) - s(A \setminus \{y\}) \geq s([a]) - s([a-1]) = s_{[a]}(a) = \left\lfloor \frac{a}{2} \right\rfloor. \quad (6)$$

Now let $b := \max(A)$. So $b \geq a+1$. We have that $s_{[b]}(b) = \lfloor b/2 \rfloor$. Now $\{1, b-1\} \cup \{2, b-2\} \cup \ldots \cup \{\lfloor b/2 \rfloor, \lceil b/2 \rceil\}$ is a partition of $[b-1]$, and each part destroys one additive triple in $[b]$ containing b. But at least $\lceil |[b] \setminus A|/2 \rceil = \lceil (b-a)/2 \rceil$ of these parts intersect with $[b] \setminus A$. Thus $s_{[b]}(b, [b] \setminus A) \geq \lceil (b-a)/2 \rceil$. Then

$$s_A(b) = s_{[b]}(b) - s_{[b]}(b, [b] \setminus A) \leq \left\lfloor \frac{b}{2} \right\rfloor - \left\lceil \frac{b-a}{2} \right\rceil = \left\lfloor \frac{a}{2} \right\rfloor. \quad (7)$$

Thus we have equality in (5)–(7) for $y = b$. So (5) implies that $s(A) = s^+(a)$ and $s(A') = s^+(a-1)$ where $A' := A \setminus \{b\}$. That is, A' is a maximising set of size $a-1$. Since a is minimal, there is some $\ell \in \mathbb{N}$ for which $A' = \ell \cdot [a-1]$, or $A' = \ell \cdot ([a-2] \cup \{a\})$ if $a-1$ is odd. Since every sum of elements in A' is a multiple of ℓ, we must have $b \in \ell \cdot \mathbb{N}$ or otherwise $s_{A'}(b) = 0$. Thus we may assume without loss of generality that $\ell = 1$.

Let G be the graph with vertex set A' in which $\{x, y\}$ is an edge whenever $\{x, y, b\}$ is an additive triple. Using the fact that $b > \max(A')$, we see that $x + y = b$ for every edge $\{x, y\}$, and G has $s_A(b) = \lfloor a/2 \rfloor \geq 2$ edges. Note that G contains at most one loop (an edge of the form $\{x, x\}$), with equality if and only if b is even and $b/2 \in A'$. The non-loop edges form a matching, so there are either $2\lfloor a/2 \rfloor$ or $2\lfloor a/2 \rfloor - 1$ non-isolated vertices. Since G has $a - 1$ vertices, it has at most one isolated vertex, with equality if and only if a is odd and there is a single loop.

First consider the case when $A' = [a - 1]$. If 1 has some neighbour x in G, then $b = x + 1 \leq a$, a contradiction. So 1 is the sole isolated vertex in G. Thus in $[2, a - 1]$, the smallest element 2 must be adjacent to the largest $a - 1$ in G, so $b = a + 1$. Thus $A = [a - 1] \cup \{a + 1\}$, which is again a contradiction.

Finally consider the case when $A' = [a - 2] \cup \{a\}$, in which case a is even. So G consists of $a/2 \geq 2$ edges and no vertex is isolated. Since $a - 1 \geq 3$, we see that $\{1, a\}$ and $\{2, a - 2\}$ must be distinct edges. So $1 + a = b = 2 + a - 2$, a contradiction. This completes the proof of (ii). ∎

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