Ringel's tree packing conjecture in quasirandom graphs

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Abstract

We prove that any quasirandom graph with n vertices and rn edges can be decomposed into n copies of any fixed tree with r edges. The case of decomposing a complete graph establishes a conjecture of Ringel from 1963.

1 Introduction

This paper concerns the following conjecture posed by Ringel [30] in 1963.

Ringel's Conjecture. For any tree T with n edges, the complete graph K_{2n+1} has a decomposition into $2n+1$ copies of T.

We prove this conjecture for large n, via the following theorem which is a generalisation to decompositions of quasirandom graphs into trees of the appropriate size. For the statement and throughout we use the following quasirandomness definition: we say that a graph G on n vertices is (ξ, s)-*typical* if every set S of at most s vertices has $((1 \pm \xi)d(G))^{|S|}n$ common neighbours, where $d(G) = e(G)\binom{n}{2}^{-1}$ is the density of G.

Theorem 1.1. *There is $s \in \mathbb{N}$ such that for all $p > 0$ there exist ξ, n_0 such that for any $n \geq n_0$ such that $p(n-1)/2 \in \mathbb{Z}$ and any tree T of size $p(n-1)/2$, any (ξ, s)-typical graph G on n vertices of density p can be decomposed into n copies of T.*

The case $p = 1$ of Theorem 1.1 establishes Ringel's conjecture for large n, a result also recently obtained independently by Montgomery, Pokrovskiy and Sudakov [28] by different methods, along the lines of their proof of an asymptotic version in [27]. They show that certain edge-colourings of K_{2n+1} contain a rainbow copy of T, such that the required T-decomposition can be obtained by cyclically shifting this rainbow copy. This approach is specific to the complete graph, and does not apply to the more general setting of quasirandom graphs as in Theorem 1.1.

Ringel's conjecture was well-known as one of the major open problems in the area of *graph packing*, whose history we will now briefly discuss. In a graph packing problem, one is given a host graph G and another graph F and the task is to fit as many edge-disjoint copies of F into G as possible. If the size (number of edges) of F divides that of G, it may be possible to find a perfect

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packing, or F-*decomposition* of G. More generally, given a family $\mathcal{F}$ of graphs of total size equal to the size of G, we seek a partition of (the edge set of) G into copies of the graphs in $\mathcal{F}$.

These problems have a long history, going back to Euler in the eighteenth century. The flavour of the problem depends very much on the size of F. The earliest results concern F of fixed size, in which case F-decompositions can be naturally interpreted as combinatorial designs. For example, Kirkman [22] showed that K_n has a triangle decomposition whenever n satisfies the necessary divisibility conditions $n \equiv 1$ or 3 mod 6; for historical reasons, such decompositions are now known as Steiner Triple Systems. Wilson [32, 33, 34, 35] generalised this to any fixed-sized graph in the 70's, and Keevash [17] to decompositions into complete hypergraphs, thus estalishing the Existence Conjecture for designs. A different proof and a generalisation to F-decompositions for hypergraphs F were given by Glock, Kühn, Lo and Osthus [12, 13]. A further generalisation that captures many other design-like problems, such as resolvable hypergraph designs (the general form of Kirkman's celebrated 'Schoolgirl Problem') was given by Keevash [18].

There is also a large literature on F-decompositions where the number of vertices of F is comparable with, or even equal to, that of G. Classical results of this type are Walecki's 1882 decompositions of K_{2n} into Hamilton paths, and of K_{2n+1} into Hamilton cycles. There are many further results on Hamilton decompositions of more general host graphs, notably the solution in [7] of the Hamilton Decomposition Conjecture, namely the existence of a decomposition by Hamilton cycles in any $2r$-regular graph on n vertices, for large n and $2r \geq \lfloor n/2 \rfloor$.

Much of the literature on F-decompositions for large F concerns decompositions into trees. Besides Ringel's conjecture, the other major open problem of this type is a conjecture of Gyárfás [14], saying that K_n should have a decomposition into any family of trees $T_1, \dots, T_n$ where each T_i has i vertices. Both conjectures have a large literature of partial results; we will briefly summarise the most significant of these (but see also [6, 9, 21, 24]). Joos, Kim, Kühn and Osthus [15] proved both conjectures for bounded degree trees. Ferber and Samotij [10] and Adamaszek, Allen, Grosu and Hladký [1] obtained almost-perfect packings of almost-spanning trees with maximum degree $O(n/\log n)$. These results were generalised by Allen, Böttcher, Hladký and Piguet [5] to almost-perfect packing of spanning graphs with bounded degeneracy and maximum degree $O(n/\log n)$. Allen, Böttcher, Clemens and Taraz [2] extended [5] to perfect packings provided linearly many of the graphs are slightly smaller than spanning and have linearly many leaves. The above results mainly use randomised embeddings, for which a maximum degree bound $O(n/\log n)$ is necessary for concentration of probability. While the results of Montgomery, Pokrovskiy and Sudakov [26, 27] mentioned above also use probabilistic methods, they are able to circumvent the maximum degree barrier by methods such as the cyclic shifts mentioned above.

Our proof proceeds via a rather involved embedding algorithm, discussed and formally presented in the next section, in which the various subroutines are analysed by a wide range of methods, some of which are adaptations of existing methods (particularly from [26] and [2], and also our own recent methods in [20] for the 'generalised Oberwolfach problem', which are in turn based on [18]), but most of which are new, including a method for allocating high degree vertices via partitioning and edge-colouring arguments and a method for approximate decompositions based on a series of matchings in auxiliary hypergraphs.

1.1 Notation

Given a graph $G = (V, E)$, when the underlying vertex set V is clear, we will also write G for the set of edges. So $|G|$ is the number of edges of G. Usually $|V| = n$. The *edge density* $d(G)$ of G is $|G|/\binom{n}{2}$.

We write $N_G(x)$ for the neighbourhood of a vertex x in G. The degree of x in G is $d_G(x) = |N_G(x)|$. For $A \subseteq V(G)$, we write $N_G(A) := \bigcap_{x \in A} N_G(x)$; note that this is the common neighbourhood of all vertices in A, not the neighbourhood of A.

We often write $G(x) = N_G(x)$ to simplify notation. In particular, if M is a matching then $M(x)$ denotes the unique vertex y (if it exists) such that $xy \in M$. We also write $M(S) = \bigcup_{x \in S} M(x)$, which is *not* consistent with our notation $N_G(S)$ for common neighbourhoods, but we hope that no confusion will arise, as we only use this notation if M is a matching, when all common neighbourhoods are empty.

We say G is (ξ, s)-*typical* if $|N_G(S)| = ((1 \pm \xi)d(G))^{|S|}n$ for all $S \subseteq V(G)$ with $|S| \leq s$.

In a directed graph J with $x \in V(J)$, we write $N_J^+(x)$ for the set of out-neighbours of x in G and $N_G^-(x)$ for the set of in-neighbours. We let $d_G^{\pm}(x) := |N_G^{\pm}(x)|$. We define common out/in-neighbourhoods $N_J^{\pm}(A) = \bigcap_{x \in A} N_J^{\pm}(x)$.

The vertex set $V(G)$ will often come with a cyclic order, identified with the natural cyclic order on $[n] = \{1, \ldots, n\}$. For any $x \in V$ we write x^+ for the successor of x, so if $x \in [n]$ then x^+ is $x+1$ if $x \neq n$ or 1 if $x = n$. Write $S^+ = \{x^+ : x \in S\}$ for $S \subseteq V(G)$. We define the predecessor x^- similarly. Given x, y in $[n]$ we write $d(x, y)$ for their cyclic distance, i.e. $d(x, y) = \min\{|x - y|, n - |x - y|\}$.

We say that an event E holds with high probability (whp) if $\mathbb{P}(E) > 1 - \exp(-n^c)$ for some $c > 0$ and $n > n_0(c)$. We note that by a union bound for any fixed collection $\mathcal{E}$ of such events with $|\mathcal{E}|$ of polynomial growth whp all $E \in \mathcal{E}$ hold simultaneously.

We omit floor and ceiling signs for clarity of exposition.

We write $a \ll b$ to mean $\forall\, b > 0\ \exists\, a_0 > 0\ \forall\, 0 < a < a_0$.

We write $a \pm b$ for an unspecified number in $[a - b, a + b]$.

2 Proof overview and algorithm

Suppose we are in the setting of Theorem 1.1: we are given an $(\xi, 2^{50 \cdot 8^3})$-typical graph G on n vertices of density p, where $n^{-1} \ll \xi \ll p$, and we need to decompose G into n copies of some given tree T with $p(n-1)/2$ edges. In this section we present the algorithm by which this will be achieved. After describing and motivating the algorithm, we present the formal statement in the next subsection, then various lemmas analysing certain subroutines over the following few subsections. We defer the analyses of the approximate decomposition to Section 3 and the exact decomposition to Section 4.

As discussed in the introduction, the most significant technical challenge not addressed by previous attempts on Ringel's Conjecture is the presence of high degree vertices, so naturally these will receive special treatment. Our algorithm will consider three separate cases for the tree T (similarly to [26]), one of which (Case L) handles trees in which almost all (i.e. all but $o(n)$) vertices belong to large stars (i.e. of size $> n^{1-o(1)}$). Case L is handled by the subroutine LARGE STARS, which will be discussed later in this overview. The other two cases for T are Case S, when T has linearly many leaves in small stars, and Case P, when T has linearly many vertices in vertex-disjoint long bare paths. In both Case S and P, we apply essentially the same 'approximate step' algorithm to embed edge-disjoint copies of $F = T \setminus P_{\text{ex}}$, obtained from T by removing the part that will be embedded in the 'exact step', so P_{ex} consists of stars in Case S and of bare paths in Case P. The overview of the proof according to these cases is illustrated by Figure 1.

The heart of the approximate step algorithm is the subroutine APPROXIMATE DECOMPOSITION, where in each step we extend our partial embeddings $(\phi_w : w \in W)$ of F by defining them on some set A_i which is suitably nice: A_i is independent, has linear size, has no vertices of degree

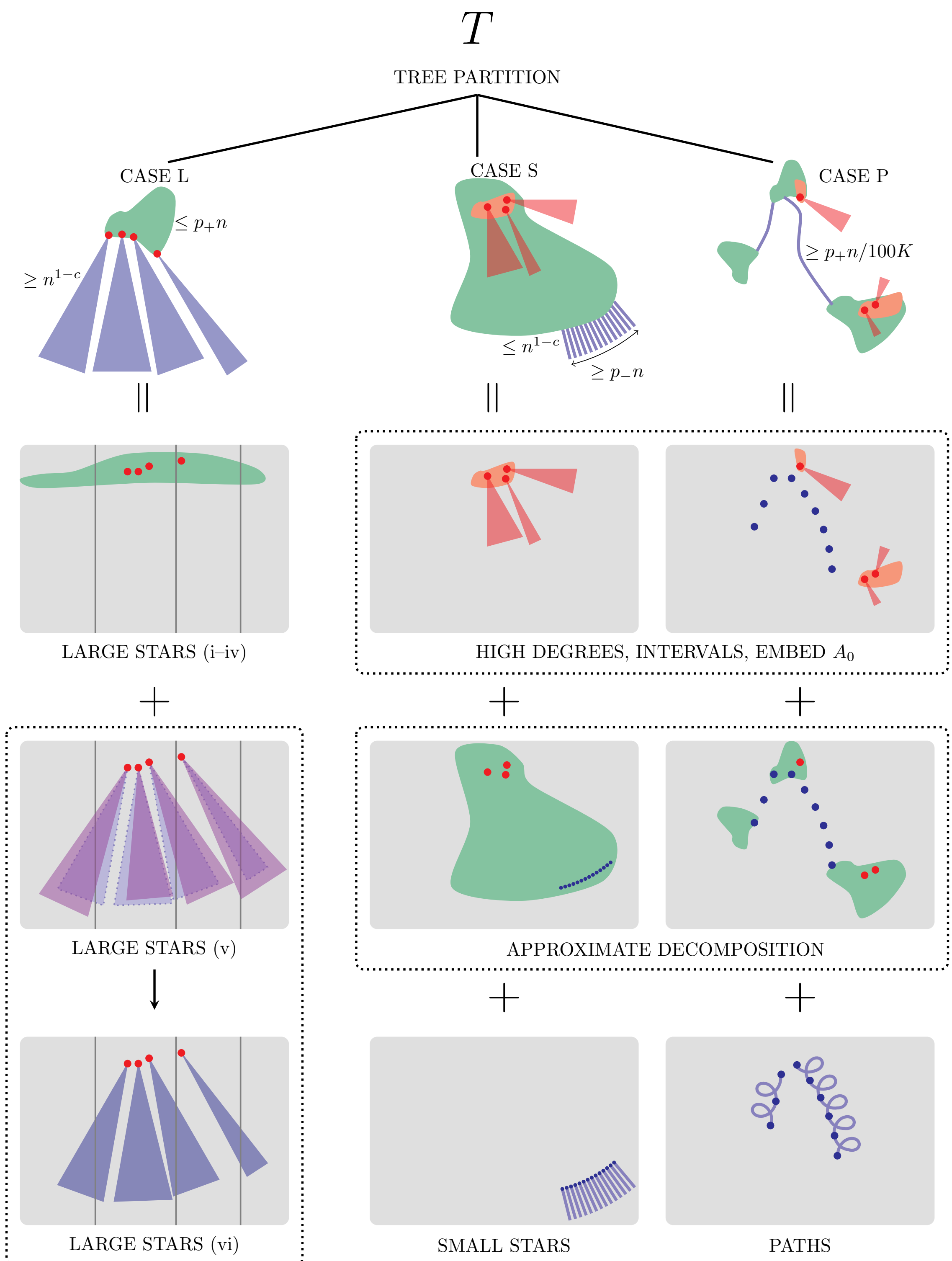

Figure 1: The three cases of the proof and the subroutines of the algorithm which embed each part of T. From left to right, Case L: almost all vertices lie in large stars; Case S: linearly many vertices lie in small stars; Case P: linearly many vertices lie in long bare paths. Red denotes high degree vertices and their neighbours. Blue denotes the part embedded in the exact step.

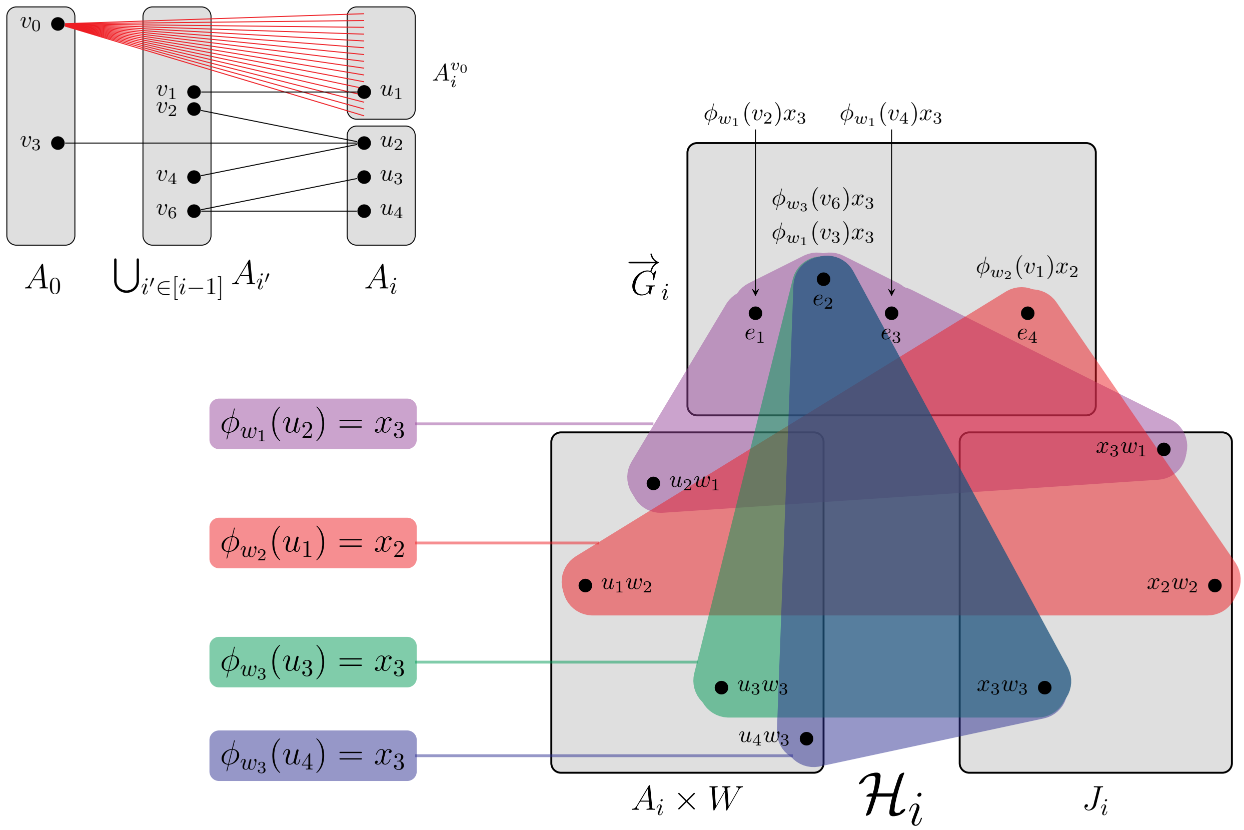

Figure 2: Part of the hypergraph $\mathcal{H}_i$, where a section of $F[A_i, A_{<i}]$ and some of the corresponding edges of $\mathcal{H}_i$ are illustrated. Here, $u_1 \in A_i^{\mathrm{hi}}$, $u_2 \in A_i^{\mathrm{lo}}$ and $u_3, u_4 \in A_i^{\mathrm{no}}$. In a previous embedding, we set $\phi_{w_1}(v_3) = \phi_{w_3}(v_6) = x_1$, and now the arc $e_2 = x_3x_1$ would be used by the potential embeddings "$\phi_{w_1}(u_2) = x_3$" (purple edge), "$\phi_{w_3}(u_3) = x_3$" (green edge) and "$\phi_{w_3}(u_4) = x_3$" (blue edge). In particular, at most one of these embeddings is allowed.

$> n^{o(1)}$, and every vertex of A_i has at most four previously embedded neighbours. We find these extensions simultaneously via a matching in an auxiliary hypergraph $\mathcal{H}_i$ (see Figure 2), which has an edge denoted "$\phi_w(u){=}x$" whenever it is possible to define "$\phi_w(u){=}x$" for some $w \in W$, $u \in A_i$, $x \in V = V(G)$. We encode the various constraints that must be satisfied by the embeddings in the definition of these edges. Thus "$\phi_w(u){=}x$" includes (as an auxiliary vertex in $V(\mathcal{H}_i)$) all arcs $\overrightarrow{yx}$ where $y = \phi_w(b)$ is a previously defined embedding of some neighbour b of a; this ensures that we maintain edge-disjointness of the embeddings of F. We also include in "$\phi_w(u){=}x$" auxiliary vertices uw and xw, to ensure that every $\phi_w(u)$ is defined at most once and ϕ_w is injective.

We ensure that $\mathcal{H}_i$ is suitably nice (its edges can be weighted so that every vertex has weighted degree $1 + o(1)$ and all weighted codegrees are $n^{-o(1)}$), in which case it is well-known from the large literature developing Rödl's semi-random 'nibble' [31], in particular [16], that one can find an almost perfect matching that is (in a certain sense) quasirandom (we use a convenient refined formulation of this statement recently presented in [8]). The quasirandomness of this matching is important for several reasons, including quasirandomness of the extensions of the embeddings to A_i, which in turn implies that later hypergraphs $\mathcal{H}_j$ with $j > i$ are suitably nice (with weaker specific parameters), and so the process can be continued.

The above sketch yields an alternative method for approximate decomposition results along the lines of those mentioned in the introduction, but has not yet dealt with high degree vertices. We will partition $V(F)$ into $A_0, A_1, \dots, A_{i^*}$, where A_i for $i \geq 1$ are the nice sets described above, and A_0 is not nice – in particular, there is no bound on the degree of vertices in A_0. We start the embedding of F in the subroutine HIGH DEGREES by embedding vertices sequentially in a suitable order, where when we consider some $a \in A_0$ we define $\phi_w(a)$ for all $w \in W$ simultaneously via a random matching $M_a = \{\phi_w(a)w : w \in W\}$ in an auxiliary bipartite graph $B_a \subseteq V \times W$, where the definition of B_a encodes constraints that must be satisfied by the embedding: we only allow an edge vw if $v \notin \operatorname{Im} \phi_w$ and v is adjacent via unused edges to all $\phi_w(b)$ where b is a previously embedded neighbour of a. (For simplicity we have suppressed several further details in the above description which will be discussed below.) The important point about this construction is that each $v \in V$ has to accommodate the vertex a for a unique embedding ϕ_w, so however large the degrees in T may be, the total demand for 'high degree edges' is the same at every vertex, and can be allocated to a digraph H which is an orientation of a quasirandom subgraph of G.

This digraph H is one of many oriented quasirandom subgraphs into which G is partitioned by the subroutine DIGRAPH, where each piece is reserved for embedding certain subgraphs of F, with arcs directed from earlier to later vertices. Besides H, these include graphs $G^{gg'}_{ii'}$ for embedding subgraphs $F'[A^g_i, A^{g'}_{i'}]$, according to a partition of each A_i into $A^{\rm hi}_i$, $A^{\rm lo}_i$, $A^{\rm no}_i$. Here $A^{\rm hi}_i$ consists of vertices adjacent to some vertex with many neighbours in A_i (which will lie in A_0 and be unique), $A^{\rm lo}_i$ consists of vertices adjacent to some vertex in A_0 (which will be unique) that does not have many neighbours in A_i, and $A^{\rm no}_i$ consists of vertices with no neighbours in A_0. To ensure concentration of probability the above sets are not defined if they would have size $o(n)$, in which case the corresponding vertices are instead added to A_0. By partitioning G in this manner we can ensure edge-disjointness when embedding different parts of F separately. To ensure injectivity of the embeddings, we also randomly partition $V \times W$ into various subgraphs in which w-neighbourhoods prescribe the allowed images in ϕ_w of the various parts of the decomposition of $V(T)$. In particular, while constructing the high degree digraph H, we also construct $J^{\rm hi}_i \subseteq V \times W$ so that each $\phi_w(A^{\rm hi}_i)$ will be approximately equal to $J^{\rm hi}_i(w)$.

The separate treatment of these parts of A_i and careful construction of A_0 to ensure the uniqueness properties mentioned above is designed to handle a considerable technical difficulty that we glossed over above when describing the embedding of A_0. Our approach to the approximate decomposition discussed above depends on maintaining quasirandomness, but we cannot ensure that $|A_0|/n$ is negligible compared with $1/i^*$, where i^* is the number of steps in the approximate decomposition, so a naive analysis will fail due to blow-up of the error terms. We therefore partition A_0 into A^*, A^{**} and A'_0, which are embedded sequentially, where $|A^*|/n$ and $|A^{**}|/n$ are negligible compared with $1/i^*$, and so do not contribute much to the error terms. For A'_0, we cannot entirely avoid large error terms, but we can confine them to a set of $o(n)$ bad vertices, via arguments based on Szemerédi regularity; these arguments require degrees in A'_0 to be bounded independently of n, so A^{**} is introduced to handle degrees that are $\omega(1)$ but $< n^{o(1)}$. The careful choice of partition ensures that these bad error terms are only incurred by vertices in $A^{\rm lo}$.

At this point, we return to consider various details glossed over in the above description of HIGH DEGREES. While the embedding via random matchings ensures that every vertex of G has the same demand of high degree edges, we also need to plan ahead when embedding $A^* \subseteq A_0$ (which contains the very high degree vertices) so that it will be possible to allocate the other ends of these edges to distinct vertices for each w, i.e. so that $\phi_w(u) \neq \phi_w(u')$ whenever $u \neq u'$. To achieve this in DIGRAPH, we randomly partition V into $(U_h : h \in [m])$, with $m = n^{1-o(1)}$, where each U_h will

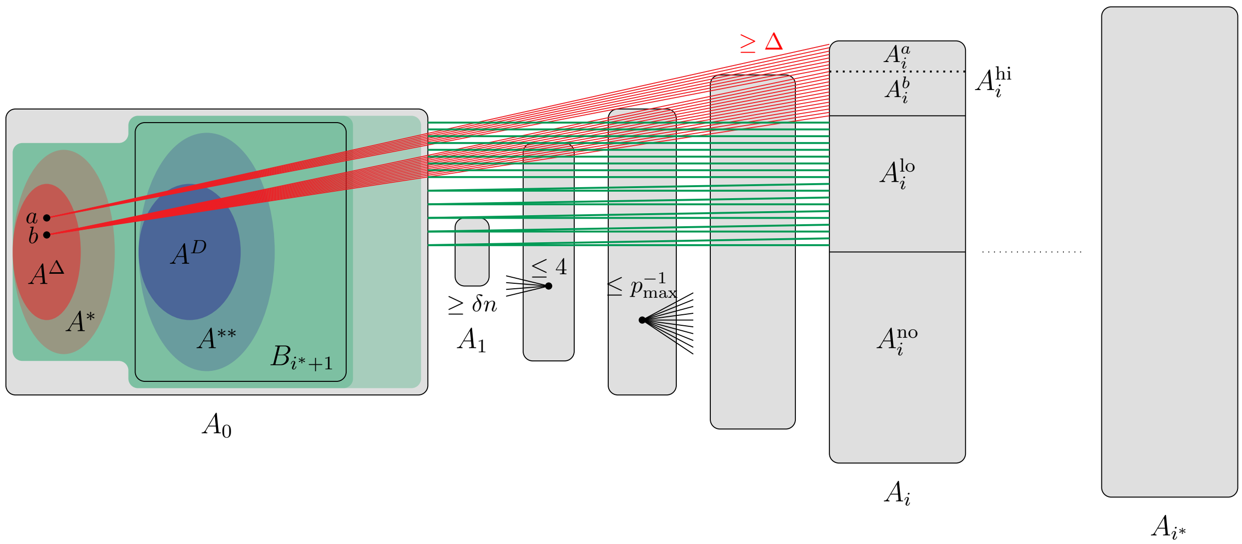

Figure 3: Partition of F obtained in TREE PARTITION. Red edges are from $T \setminus F$ and green edges are F-edges from A_0 to A_i (so A_i^{lo}).

accommodate those ends of high degree edges corresponding to colour h in a certain properly m-edge-coloured bipartite multigraph in $V \times W$, i.e. $\overrightarrow{yx}$ is available for H if yw has colour h and $x \in U_h$. Thus $x = \phi_w(u) \in U_h$ and $x' = \phi_w(u') \in U_{h'}$ are distinct automatically if $h \neq h'$, and due to properness of the colouring if $h = h'$, as c determines a unique $y \in V(G)$, so a unique $a = \phi_w^{-1}(y) \in A^*$.

The above multigraph M in $V \times W$ consists of copies of $M_a^* \approx M_a$ for each $a \in A^*$, with the copies distinguished by labels ℓ_{aij}, where for each $a \in A^*$ and part A_i in which a has many neighbours the number of labels ℓ_{aij} is proportional to the degree of a in A_i. An edge yw of label ℓ_{aij} in M^h means that H arcs $\overrightarrow{yx}$ with $x \in U_h$ will be allocated to edges au of F with $a = \phi_w^{-1}(y)$ and $u \in N_F(a) \cap A_i$. For typicality we require for any a and i that the number of edges in each M^h with some label ℓ_{aij} is approximately independent of h.

This is achieved by a construction based on cyclic shifts, which we will now sketch, suppressing some details. We partition V into V_0 and $(V_{v^*} : v^* \in V^*)$ and W into W_0 and $(W_{w^*} : w^* \in W^*)$, where V_0 and W_0 are small, V^* and W^* are copies of $[m]$, and all V_{v^*}, W_{w^*} have the same size. The matchings M_a are chosen as $M_a^0 \cup M_a^*$, where $V(M_a^0) = V_0 \cup W_0$ and if $vw \in M_a^*$ then $v \in V_{v^*}$, $w \in W_{w^*}$ with $v^* = x_a + w^*$, according to some cyclic shifts $(x_a : a \in A^*)$, carefully chosen to ensure edge-disjointness. We construct a labelled multigraph in $V^* \times W^*$ analogously to that in $V \times W$, and obtain label-balanced matchings M^h for all $h \in [m]$ as cyclic shifts of some fixed label-balanced matching M' in $V^* \times W^*$, where for each $v^*w^* \in M'$ with some label ℓ_{aij} we include in M^h all edges of M_a^* of the same label between V_{v^*+h} and W_{w^*+h}.

The above description of M^h is over-simplified, as in fact we construct two such matchings, one handling vertices of huge degree (almost linear) and the other handling vertices with degree that is high but not huge. The version of M' for non-huge degrees is constructed by the same hypergraph matching methods as in the above description of the approximate step embeddings, but these do not apply to huge degrees (the codegree bound fails) so we instead apply a result of Barát, Gyárfás and Sárközy [3] on rainbow matchings in properly coloured bipartite multigraphs. The construction is illustrated in Figure 4.

The exact steps in Cases S and P are handled by adapting existing methods in the literature. In Case P, the subroutines INTERVALS and PATHS are adaptations of the methods we used in [20] for the 'generalised Oberwolfach Problem' of decomposing any quasirandom even regular oriented graph into prescribed cycle factors; we refer the reader to this paper for a detailed discussion of these methods. In Case S, we find the required stars by adapting an algorithm of [2]: we find an orientation of the unused graph so that the outdegree of each vertex is precisely the total size of stars it requires in all copies of T, and then process each vertex in turn, using random matchings to partition its outneighbourhood into stars of the correct sizes, while maintaining injectivity of the embeddings.

It remains to consider the exact step in Case L, when almost every vertex of T is a leaf adjacent to a vertex of very large degree; this is more challenging and requires new methods (the arguments used in Case S fail due to lack of concentration of probability). The most difficult constraint to satisfy is injectivity of the embeddings, so we build this into the construction explicitly: we randomly partition $V(G)$ into sets U^a for each star centre a and require each embedding to choose most of its leaves for its copy of a within U^a. Each edge xy of G, say with $x \in U^a$, $y \in U^b$, will be randomly allocated one of two options: (i) x is a leaf of a star in some embedding ϕ_w with $\phi_w(a) = y$, or (ii) y is a leaf of a star in some embedding $\phi_{w'}$ with $\phi_{w'}(b) = x$. A final balancing step will swap edges between stars (thus slightly bending the rules on leaf allocation) so that all stars are exactly as required; see Figure 5. The above sketch can be implemented for decomposing a quasirandom graph into star forests, but there is a considerable extra difficulty caused by the constraints imposed by the initial embedding of the small part of T not contained in the large stars.

A naive approach to this embedding can easily cause many edges of G to be unusable according to the rules for U^a as described above. Indeed, for each edge xy of G, the two options as described above will both become unavailable during the initial embedding if we choose both $\phi_w(a') = x$ for some a' and $\phi_{w'}(b') = y$ for some b'. We therefore keep track of a digraph J that records these constraints and choose the initial embedding so that each edge of G always has at least one of its two options available. To control these constraints, we also introduce partitions of each U^a into three parts, and also of the set W indexing the embeddings into three parts, and impose two different patterns for matching parts of U^a with parts of W according to whether or not a vertex has large degree. The digraph J and its use in defining available sets for the embedding are illustrated in Figure 6.

2.1 Formal statement of the algorithm

The input to the algorithm consists of a (ξ, s)-typical graph G on n vertices of density p, where $s = 2^{50 \cdot 8^3}$, $n^{-1} \ll \xi \ll p$, and a tree T with $p(n-1)/2$ edges. We fix $0 < c' \ll c \ll 1$ and parameters

$$n^{-1} \ll \xi \ll \eta_- \ll p_- \ll \eta_+ \ll p_+ \ll p, \qquad \text{and } \Delta = n^c,\ \Lambda = n^{1-c}.$$

Recall that a *leaf* in T is a vertex of degree 1 in T. We call an edge a *leaf edge* if it contains a leaf. We call a star a *leaf star* if it consists of leaf edges. We call a path in T a *k-path* if it has length k (that is, k edges), and call it *bare* if its internal vertices all have degree 2 in T. By Lemma 2.9 below we can choose a case for T in {L,S,P} satisfying

- Case L: all but at most $p_+ n$ vertices of T belong to leaf stars of size $\geq \Lambda$,
- Case S: at least $p_- n$ vertices of T belong to leaf stars of size $\leq \Lambda$,
- Case P: T contains $p_+ n/100K$ vertex-disjoint bare $8K$-paths.

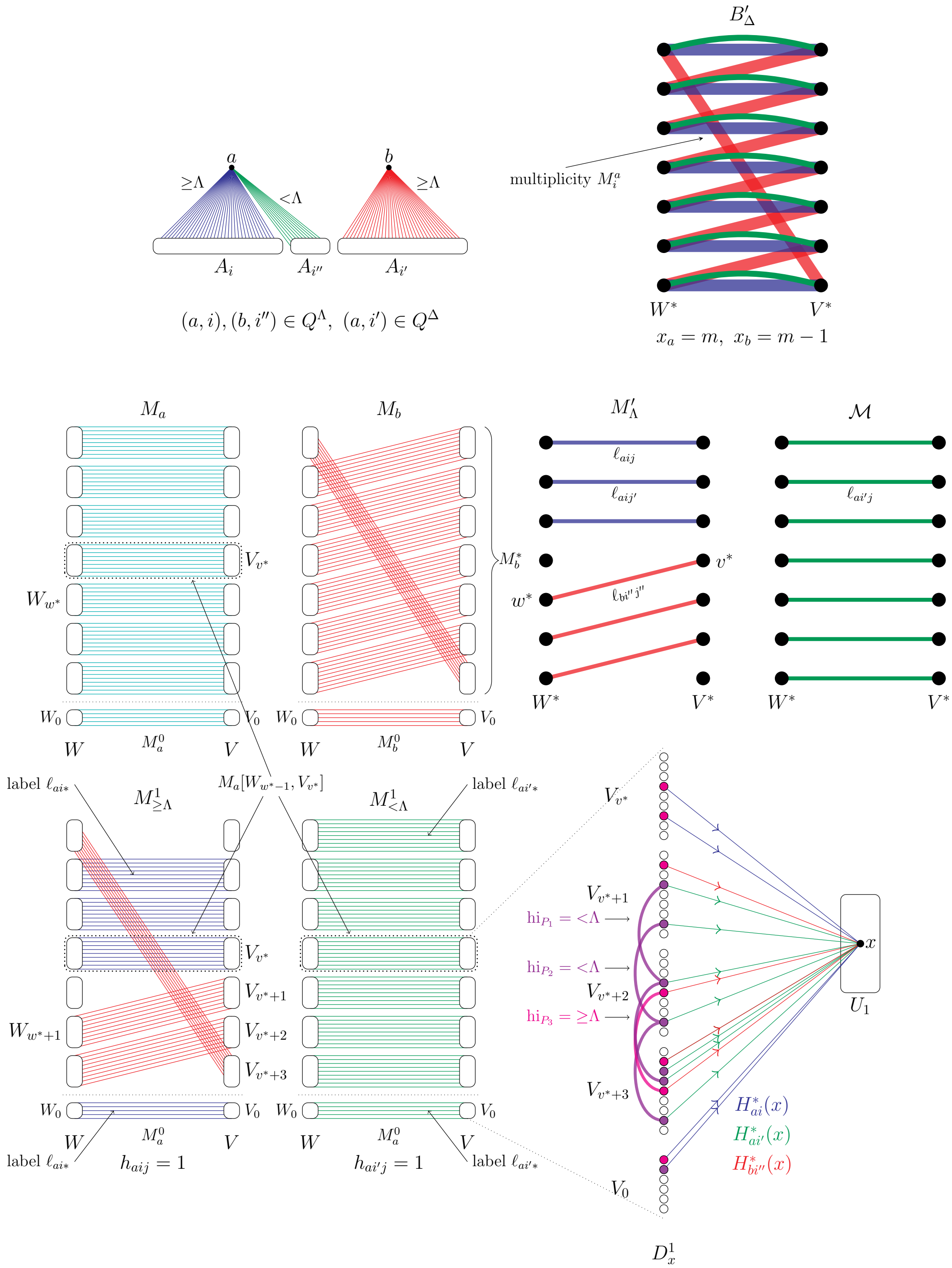

Figure 4: From left to right, top to bottom: two high degree vertices a, b; the multigraph B'_Δ where line thickness represents multiplicity; the matchings M_a, M_b between W and V; the matchings $M'_\Lambda, \mathcal{M}$ on W^*, V^*; the matchings $M^1_{\geq\Lambda}, M^1_{<\Lambda}$; the graph D^1_x for $x \in U_1$ with components P coloured to represent the random choice $\mathrm{hi}_P \in \{\geq\Lambda, <\Lambda\}$; the resulting edges of H^*_{ai} at x.

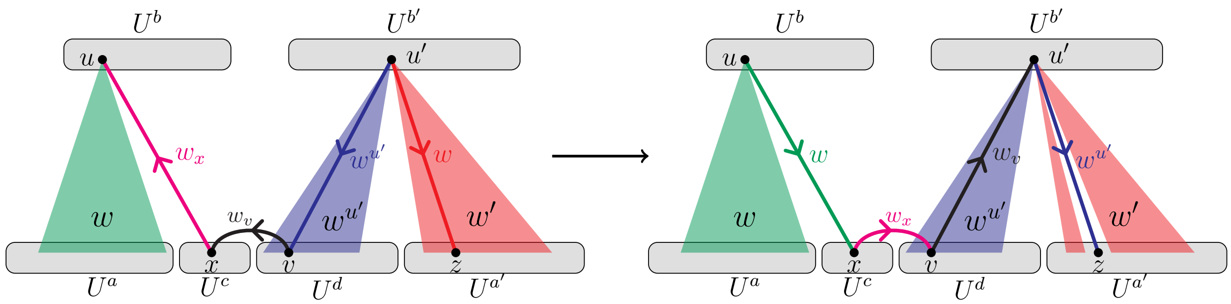

Figure 5: A single step of the algorithm to modify the green star which is too small and the red star which is too large.

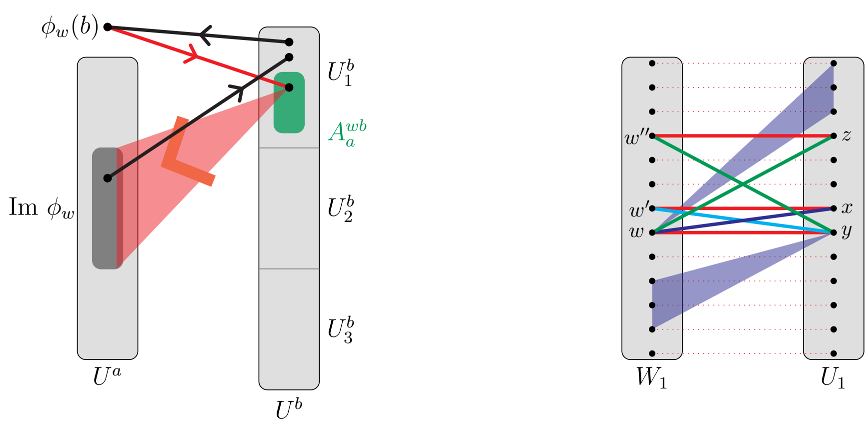

Figure 6: (Left) the available set A_a^{wb} for $w \in W_1$ and $a \in S$. The black arcs are some arcs in J; they forbid their U^b-endvertices from A_a^{wb}. The red arcs would be added to J if the labelled vertex is chosen for $\phi_w(a)$. (Right) A pair of edges wy, $w'x$ that must be avoided by the matching defining the embeddings of a, and a swap that may be implemented by Lemma 2.7 to remove wy. Red edges define images of a and blue edges define images of some other vertices.

In Case L go to LARGE STARS, otherwise continue. We let $\eta = \eta_-$ in Case S or $\eta = \eta_+$ in Case P, and define further parameters

$$\xi \ll \xi' \ll D^{-1} \ll \delta \ll p_{\min} \ll \varepsilon_1 \ll \ldots \ll \varepsilon_{i^+} \ll p_{\max} \ll \varepsilon \ll p_0 \ll \eta \ll s^{-1}, p,$$

with $i^+ = 7 \log \varepsilon^{-1}$ and $\xi' \ll K^{-1} \ll d^{-1} \ll D^{-1}$ in Case P. Given $k \in \mathbb{N}$, a tree T and $S \subseteq V(T)$, the *k-span* $\mathrm{span}^k_T(S)$ of S in T is obtained by starting with $S^* = S$ and iteratively adding any $S' \subseteq V(T) \setminus S^*$ with $|S'| \in [k]$ such that $T[S^* \cup S']$ has fewer components than $T[S^*]$, until there is no such S'. Clearly there are at most $|S|$ iterations, so $|\mathrm{span}^k_T(S)| \leq (k+1)|S|$. Note also that $|\mathrm{span}^k_T(\mathrm{span}^k_T(A) \cup B) \setminus \mathrm{span}^k_T(A)| \leq (k+1)|B|$. For $k \in \mathbb{N}$ let $A^k = \{u : d_T(u) \geq k\}$.

TREE PARTITION

i. Let $A^* = \mathrm{span}^4_T(A^\Delta)$. In Case S let P_{ex} be a union of leaf stars in $T \setminus T[A^*]$, each of size $\leq \Lambda$, with $|P_{\mathrm{ex}}| = p_- n/2 \pm \Lambda$. In Case P let P_{ex} be the vertex-disjoint union of two leaf edges in $T \setminus T[A^*]$ and $p_+ n/101K$ bare $8K$-paths in $T \setminus T[A^*]$.
Obtain F from T by deleting all edges of P_{ex} and F^* from F by deleting all vertices of A^*.

ii. Define disjoint independent sets $C_1, \ldots, C_{i^*}$ in F^* as follows. At step $i \geq 1$, let $B_i = V(F^*) \setminus \bigcup_{j<i} C_j$, let C'_i be the set of $v \in B_i$ with $d_{F^*[B_i]}(v) \leq 3$ and $d_{F^*[\bigcup_{j<i} C_j]}(v) \leq p^{-1}_{\max}$, and let C_i be a maximum independent set in $F^*[C'_i]$. If $|C_i| < \varepsilon n$ let $i^* = i - 1$ and stop, otherwise go to the next step.

iii. Let $A_0 = \mathrm{span}^4_T[A^* \cup B_{i^*+1}]$. Let $A^{**} = \mathrm{span}^4_T(A^D) \setminus A^*$ and $A'_0 = A_0 \setminus (A^* \cup A^{**})$.
For $i \in [i^*]$ let $A_i = C_{i^*+1-i} \setminus A_0$ and for $k \in \mathbb{N}$ let $A^k_i = \{a \in A^k : |N_F(a) \cap A_i| \geq \Delta\}$.
For $a \in A^\Delta_i$ let $A^a_i = N_F(a) \cap A_i$. Let $A^{\mathrm{hi}}_i = \bigcup_{a \in A^\Delta_i} A^a_i$ and $A^{<\Lambda}_i = \bigcup_{a \in A^\Delta_i \setminus A^\Lambda_i} A^a_i$ and $A^{\geq \Lambda}_i = \bigcup_{a \in A^\Lambda_i} A^a_i$. Obtain F' from F by deleting all edges ab with $a \in A^\Delta_i$ and $b \in A^a_i$ for some i. Let $A^{\mathrm{lo}}_i = \{u \in A_i : |N_{F'}(u) \cap A_0| = 1\}$. Let $A^{\mathrm{no}}_i = \{u \in A_i : N_F(u) \cap A_0 = \emptyset\}$.

iv. For $j \in [4]$, let $\delta_j = \delta^{.1j+.6}$ and let $(\circ_1, \ldots, \circ_4) = (\mathrm{no}, \geq\Lambda, <\Lambda, \mathrm{lo})$. For each $j \in [4]$, while any $|A^{\circ_j}_i| < \delta_j n$ move $A^{\circ_j}_i$ to A_0, let $A_0 = \mathrm{span}^4_T[A_0 \cup A^{\circ_j}_i]$, and update $A^{<\Lambda}_i, A^{\geq\Lambda}_i, A^{\mathrm{lo}}_i, A^{\mathrm{no}}_i$.

v. If $\bigcup_i A^{\mathrm{hi}}_i = \emptyset$ move A^* to A^{**}, i.e. redefine A^{**} as $A^{**} \cup A^*$ and A^* as $\emptyset$.

Let $A^{\mathrm{hi}} = \bigcup_i A^{\mathrm{hi}}_i$ and define $A^{\mathrm{no}}, A^{\mathrm{lo}}$ similarly.

For $k \in \mathbb{N}$, let $Q^\Delta = \{(a, i) : \Delta \leq |N_F(a) \cap A_i| < \Lambda\} \subseteq A^\Delta \times [i^*]$ and $Q^\Lambda = \{(a, i) : |N_F(a) \cap A_i| \geq \Lambda\} \subseteq A^\Lambda \times [i^*]$. We introduce parameters

$$m^a_i = \lceil \Delta^{-.2} |A^a_i| \rceil \, 1_{a \in A^\Delta_i}, \quad m_a = \sum_i m^a_i, \quad m = \sum_{a \in A^\Delta} m_a.$$

Let $\prec$ be an order on $V(T)$ with $A^* \prec A^{**} \prec A'_0 \prec A_1 \prec \cdots \prec A_{i^*} \prec V(P_{\mathrm{ex}}) \setminus V(T)$ and $|N_<(v) \cap X| \leq 1$ whenever $v \in X \in \{A^*, A^{**}, A'_0\}$. For $v \in V(T)$ we let $<v = \{u : u \prec v\}$, $N_<(v) = N_{F'}(v) \cap <v$, $N_\leq(v) = N_<(v) \cup \{v\}$ and $N_>(v) = N_{F'}(v) \setminus <v$.

We stress the use of F' in this notation, which ensures that $N_>(a) \cap A^{\mathrm{hi}}_i = \emptyset$ for all $a \in A_0$: otherwise we would have a vertex not in A_0 adjacent to two vertices of A_0, but this contradicts the definition of A_0 as a span. We list here some other immediate consequences of the definition of A_0 that will often be used without comment.

- $|A^*| \leq 5n/\Delta$ and $|A^{**}| \leq 5n/D$.
- Any $u \in A_{\geq 1}$ has $|N_<(u) \cap A_0| \leq 1$.
- Any $uv \in F[A_{\geq 1}]$ has $|(N_<(u) \cup N_<(v)) \cap A_0| \leq 1$.
- There is no $\leq$3-path in $T \setminus A_0$ with both ends in $A^{\mathrm{hi}} \cup A^{\mathrm{lo}}$.

We also note that $|N_>(v)| \le p_{\max}^{-1}$ for all $v \in A_{\ge 1}$, and $|N_<(v)| \le 4$ for all $v \in V(T)$. To see the latter, note that if $v \in A_{\ge 1}$ then v has at most 3 earlier neighbours in $A_{\ge 1}$ and at most one in A_0, whereas if $v \in A_0$ then v has at most one earlier neighbour in each of A^*, A^{**} and A_0'.

Write $n = mn_* + n_0$ with $|n_0 - n\Delta^{-.1}| < m$. Recall that we adopt the natural cyclic orders on $[m]$ and $[n]$, addition wraps, and $d(\cdot,\cdot)$ is cyclic distance. Whenever an algorithm is required to make a choice, it aborts if it is unable to do so (we will show whp it does not abort).

Given bipartite graphs $B, Z \subseteq X \times Y$ with $|X| = |Y|$ we write M=MATCH(B, Z) to mean that M is a random perfect matching from Lemma 2.7. (The choice of Z will ensure edge-disjointness of the embeddings.)

HIGH DEGREES

i. Choose $x_a \in [m]$ for $a \in A^*$ in $\prec$ order, arbitrarily subject to $d(x_a, x_{a'}) > 3d$ for all $a' \prec a$, and $d(x_a, x_{a'}) \ne d(x_b, x_{b'})$ for all $a' \in N_<(a)$ and $bb' \in F[<a]$.
ii. Choose independent uniformly random partitions of $V(G)$ into V_0 of size n_0 and V_{v^*}, $v^* \in V^*$ of size n_*, and W into W_0 of size n_0 and W_{w^*}, $w^* \in W^*$ of size n_*, where $V^* = W^* = [m]$.
iii. For each $a \in A^*$ in $\prec$ order we will define all $\phi_w(a)$ by choosing a perfect matching $M_a = \{\phi_w(a)w : w \in W\}$. Let $B_a \subseteq V \times W$ consist of all vw where $v \notin \operatorname{Im}\phi_w$ and each $\phi_w(b)v$ with $b \in N_<(a)$ is an unused edge of G. Let $Z_a \subseteq V \times W$ consist of all $\phi_w(b)w$ with $b \in N_<(a)$.
Let $B_a^0 = B_a[V_0, W_0]$ and $B_a^{w^*} = B_a[V_{x_a + w^*}, W_{w^*}]$ for $w^* \in W^*$. Define Z_a^0 and $Z_a^{w^*}$ similarly.
Let $M_a = M_a^0 \cup M_a^*$ with M_a^0=MATCH(B_a^0, Z_a^0) and M_a^*=$\bigcup_{w^*}$MATCH$(B_a^{w^*}, Z_a^{w^*})$.

We randomly identify $V(G)$ with $[n]$, cyclically ordered as above. Recall that each $x \in [n]$ has successor $x^+ = x+1$ (where $n+1$ means 1) and predecessor $x^- = x-1$ (where 0 means n). Let $d_i = d/(2s)^{i-1}$ for $i \in [2s+1]$. We write $n = r_i d_i + s_i$ with $r_i \in \mathbb{N}$ and $0 \le s_i < d_i$, and let

$$P_j^i = \begin{cases} \{kd_i + j : 0 \le k \le r_i\} & \text{if } j \in [s_i], \\ \{kd_i + j : 0 \le k \le r_i - 1\} & \text{if } j \in [d_i] \setminus [s_i]. \end{cases}$$

For each $i \in [s+1]$ and $j \in [d_i]$ we define a partition of $[n]$ into a family of cyclic intervals $\mathcal{I}_j^i$ defined as all $[x, y^-]$ where $x \in P_j^i$ and y is the next element of P_j^i in the cyclic order. (So $|\mathcal{I}_j^i| = n/d_i \pm 1$, each $I \in \mathcal{I}_j^i$ has $|I| \le d_i$, and $\mathcal{I}_j^i \cap \mathcal{I}_{j'}^i = \emptyset$ for $j \ne j'$.) We let $\mathcal{I}^i = \bigcup_{j \in [d_i]} \mathcal{I}_j^i$. (So for every $z \in [n]$, exactly one $[x, y^-] \in \mathcal{I}^i$ has $x = z$, and exactly one $[x, y^-] \in \mathcal{I}^i$ has $y = z$.)

INTERVALS

i. In Case S let $\overline{X}_w = V \setminus \phi_w(A^*)$, $\overline{p}_w := n^{-1}|\overline{X}_w|$ for all $w \in W$ and go to DIGRAPH; otherwise (in Case P) continue. For each $w \in W$ independently choose $i(w) \in [2s+1]$ and $j(w) \in [d_{i(w)}]$ uniformly at random. Let $W_i = \{w : i(w) = i\}$.
ii. For each $w \in W$, let $\mathcal{A}_w$ include each interval of $\mathcal{I}_{j(w)}^{i(w)}$ independently with probability 1/2.
Let $\mathcal{S}_w$ consist of all $I \in \mathcal{A}_w$ such that both neighbouring intervals $I^\pm$ of I are not in $\mathcal{A}_w$.
iii. For each $w \in W$, let $\mathcal{X}_w$ include each $I \in \mathcal{S}_w$ with probability $(1-\eta)n^{-1}|P_{\rm ex}|$ independently, let $X_w = \bigcup \mathcal{X}_w$, $\overline{X}_w = V \setminus (\phi_w(A^*) \cup X_w \cup (X_w)^+)$ and $\overline{p}_w = n^{-1}|\overline{X}_w|$.
iv. Obtain $\mathcal{Y}_w \subseteq \mathcal{X}_w$ as follows. Remove any I from $\mathcal{X}_w$ that intersects $\phi_w(A^*)$, let $t_i = \min\{|\mathcal{X}(I)| : I \in \mathcal{I}^i\}$, where $\mathcal{X}(I) := \{w \in W_i : I \in \mathcal{X}_w\}$, then delete each $I \in \mathcal{I}^i$, $i \in [2s+1]$ from $|\mathcal{X}(I)| - t_i$ sets $\mathcal{X}_w$ with $w \in \mathcal{X}(I)$, independently uniformly at random.
Let $Y_w = \bigcup \mathcal{Y}_w$ and $\mathcal{Y}(I) = \{w \in W_i : I \in \mathcal{Y}_w\}$.

EMBED A_0

i. For each $xy \in G^* := G \setminus \bigcup_w \phi_w(T[A^*])$ independently let $\mathbb{P}(xy \in G_0) = p_0/p$.
For each $w \in W$ and $x \in \overline{X}_w$ independently let $\mathbb{P}(\overrightarrow{xw} \in J_0) = p_0/\overline{p}_w$.

ii. Extend the embeddings ϕ_w of $T[A^*]$ to $T[A_0]$ in $\prec$ order, where for each $a \in A_0 \setminus A^*$ we choose a perfect matching $M_a = \{\phi_w(a)w : w \in W\}$=MATCH$(B_a, Z_a)$, where $Z_a = \{\phi_w(b)w : b \in N_<(a)\}$ and $B_a \subseteq V \times W$ consists of all vw with $v \in N_{J_0}(w) \setminus \operatorname{Im} \phi_w$ where each $\phi_w(b)v$ with $b \in N_<(a)$ is an unused edge of G_0.

For $i, i' \in [i^*]$ and $g, g' \in \{\text{hi}, \text{lo}, \text{no}\}$, let $p^{gg'}_{ii'} = n^{-1}|F'[A^g_i, A^{g'}_{i'}]| + p_{\min}$ and $p^g_{i0} = n^{-1}|F'[A^g_i, A_0]| + p_{\min}$. We also write $p^{gg'}_{i0} = p^g_{i0}$ for all g' for uniform notation later.

For $i \in [i^*]$, $g \in A^\Delta_i \cup \{\text{lo}, \text{no}\}$ let $\alpha^g_i = |A^g_i|n^{-1}$, $\alpha_{\text{lo}} = |A^{\text{lo}}|n^{-1}$, $\alpha_{\text{no}} = |A^{\text{no}}|n^{-1}$ and $\alpha_0 = |A_0|n^{-1}$. Let $\alpha^{\text{hi}}_i = \Delta^{.2} m_i/n$ and $\alpha_{\text{hi}} = \sum_i \alpha^{\text{hi}}_i = \Delta^{.2} m/n = |A^{\text{hi}}|n^{-1} \pm \Delta^{-.9}$.

Let $p_{\text{ex}} = n^{-1}|P_{\text{ex}}|$. Let $p'_{\text{ex}} = (\frac{7}{8} - \eta)p_{\text{ex}}$ in Case P or $p_{\text{ex}} \ll p'_{\text{ex}} \ll 1$ in Case S.

We note some identities and estimates for our parameters:

$$
\begin{gathered}
p(n-1)/2 = |T| = |T[A_0]| + |F'| + |A^{\text{hi}}| + |P_{\text{ex}}|, \\
1 + p(n-1)/2 = |V(T)| = |V(F)| + |V(P_{\text{ex}}) \setminus V(F)|, \\
\sum_{i,i',g,g'} p^{gg'}_{ii'} - n^{-1}|F'| \in [0, p_{\min}{}^{.9}], \quad \sum_i \alpha^g_i = n^{-1}|A^g|, \\
p/2 - \sum p^{gg'}_{ii'} - \alpha_{\text{hi}} - p_{\text{ex}} \in [0, p_{\min}{}^{.9}], \quad p/2 - (\alpha_{\text{hi}} + \alpha_{\text{lo}} + \alpha_{\text{no}} + p_{\text{ex}}) \in [0, \varepsilon^{.9}], \\
\overline{p}_w = \begin{cases} 1 - |A_0|n^{-1} \text{ in Case S, or} \\ \overline{p} \pm d^{-.9} \text{ with } \overline{p} = (1-\alpha_0)(1-(1-\eta)p_{\text{ex}}/8) \text{ in Case P.} \end{cases}
\end{gathered}
$$

These estimates imply that the assignment of probabilities to mutually exclusive events in DIGRAPH.vii below are valid (i.e. have sum ≤ 1). For $\circ \in \{<\Lambda, \ge\Lambda\}$, let

$$
m_\circ = \sum_{(a,i) \in Q^\circ} m^a_i, \quad p_\circ = m_\circ/m, \quad \text{and define}
$$

$$
\text{labels } L_\circ = \{\ell_{aij} : (a,i) \in Q^\circ, j \in [M^a_i]\}, \text{ where } M^a_i \in \{\lfloor m^a_i/p_\circ \rfloor, \lceil m^a_i/p_\circ \rceil\} \quad \text{and } |L_\circ| = m.
$$

DIGRAPH

i. For each $a \in A^\Delta$ let M'_a denote the perfect matching between V^* and W^* consisting of all v^*w^* with $w^* \in W^*$ and $v^* = x_a + w^* \in V^*$. Let B'_{ai} be the bipartite multigraph formed by M^a_i copies of M'_a labelled by ℓ_{aij}, $j \in [M^a_i]$. For $k \in \mathbb{N}$ let $B'_k = \bigcup_{(a,i) \in Q^k} B'_{ai}$ and B_k be the bipartite multigraph formed by M^a_i copies of M^*_a for each $(a,i) \in Q^k$ labelled by ℓ_{aij}, $j \in [M^a_i]$.

ii. Let M'_Λ be a largest matching in B'_Λ with at most one edge of each label. Define a partial m-edge-colouring $(M^h_{\ge\Lambda} : h \in [m])$ of B_Λ, where for each $h \in [m]$ and edge v^*w^* of M'_Λ with some label ℓ_{aij} we include in $M^h_{\ge\Lambda}$ all edges of M^*_a with label ℓ_{aij} between V_{v^*+h} and W_{w^*+h}.

iii. Let $(\mathcal{H}, \omega)$ be the weighted 3-graph where for each v^*w^* labelled ℓ_{aij} with $(a,i) \in Q^\Delta$ we include $v^*w^*\ell_{aij}$ with weight m^{-1}. Let $\mathcal{M}$ be a random matching obtained from Lemma 2.8 applied to $(\mathcal{H}, \omega)$. Define matchings $M^h_{<\Lambda} \subseteq B_\Delta$ for $h \in [m]$, where for each edge $v^*w^*\ell_{aij}$ of $\mathcal{M}$ we include in $M^h_{<\Lambda}$ all edges of M^*_a with label ℓ_{aij} between V_{v^*+h} and W_{w^*+h}.

iv. Partition V as $(U_h : h \in [m])$ uniformly at random. Fix distinct $h_{aij} \in [m]$ for each $\ell_{aij} \in L_\circ$ and $\circ \in \{<\Lambda, \ge\Lambda\}$ (recalling $|L_\circ| = m$). For all $\ell_{aij} \in L_\circ$, add a copy of M^0_a with every edge labelled ℓ_{aij} to $M^{h_{aij}}_\circ$. Let $p_1 = p - p_0$ and $\overrightarrow{G}_1$ be a uniformly random orientation of $G_1 := G^* \setminus (G_0 \cup \{xy : d(x,y) \ge 3d\})$.

v. For $h \in [m]$ and $x \in U_h$ let D^h_x be the graph on $N^-_{\overrightarrow{G}_1}(x)$ consisting of all yy' with $y \neq y'$ such that $yw \in M^h_{\geq\Lambda}$ and $y'w \in M^h_{<\Lambda}$ for some $w \in W$ with $x \in \overline{X}_w$. For each connected component P of D^h_x independently choose one of $\mathbb{P}(\mathrm{hi}_P = \circ) = p_\circ$ for $\circ \in \{\geq\Lambda, <\Lambda\}$. For each $y \in P$ with some $yw \in M^h_{\mathrm{hi}_P}$ with label ℓ_{aij} include $\overrightarrow{yx}$ in H^*_{ai} and let $w(\overrightarrow{yx}) = w$.

vi. For each $\overrightarrow{yx} \in \overrightarrow{G}_1$ independently choose at most one of $\mathbb{P}(\overrightarrow{yx} \in \overrightarrow{G}_{\mathrm{ex}}) = 2p_{\mathrm{ex}}/p_1$ or $\mathbb{P}(\overrightarrow{yx} \in \overrightarrow{G}^{gg'}_{ii'}) = 2p^{gg'}_{ii'}/p_1$ for $1 \leq i' < i \leq i^*$ and $g, g' \in \{\mathrm{hi}, \mathrm{lo}, \mathrm{no}\}$, or $\mathbb{P}(\overrightarrow{yx} \in \overrightarrow{G}^g_{i0}) = 2p^g_{i0}/p_1$ for $i \in [i^*]$ and $g \in \{\mathrm{hi}, \mathrm{lo}, \mathrm{no}\}$, or $\mathbb{P}(\overrightarrow{yx} \in \overrightarrow{G}'_i) = 2p_{\max}/p_1$ for $i \in [i^*]$, or $\mathbb{P}(\overrightarrow{yx} \in H) = 2\alpha_{\mathrm{hi}}/p_1\overline{p}_w$ if $x \in \overline{X}_w$, where $w = w(\overrightarrow{yx})$, and if $\overrightarrow{yx} \in H^a_i := H \cap H^*_{ai}$ include $\overrightarrow{xw} \in J^a_i$. Let $J^{\mathrm{hi}}_i = \bigcup_{a \in A^\Delta_i} J^a_i$ and $J^{\mathrm{hi}} = \bigcup_i J^{\mathrm{hi}}_i$ and $\overrightarrow{G}_i = \bigcup_{g,g',j} \overrightarrow{G}^{gg'}_{ij}$.

vii. Let J' be the set of $\overrightarrow{xw} \notin J_0 \cup J^{\mathrm{hi}}$ with $x \in \overline{X}_w$. For $\overrightarrow{xw} \in J'$ let $p_{xw} = \overline{p}_w p_1 - 2\alpha_{\mathrm{hi}}\,\mathrm{hi}_{xw}$, where hi_{xw} is 1 if $w = w(\overrightarrow{yx})$ for some y or 0 otherwise. For each $\overrightarrow{xw} \in J'$ independently choose at most one of $\mathbb{P}(\overrightarrow{xw} \in J_{\mathrm{ex}}) = p'_{\mathrm{ex}}/p_{xw}$ or $\mathbb{P}(\overrightarrow{xw} \in J^{\mathrm{lo}}_i) = \alpha^{\mathrm{lo}}_i/p_{xw}$ or $\mathbb{P}(\overrightarrow{xw} \in J^{\mathrm{no}}_i) = \alpha^{\mathrm{no}}_i/p_{xw}$ or $\mathbb{P}(\overrightarrow{xw} \in J'_i) = p_{\max}/p_{xw}$. Let $J^{\mathrm{lo}} = \bigcup_i J^{\mathrm{lo}}_i$ and $J^{\mathrm{no}} = \bigcup_i J^{\mathrm{no}}_i$.

viii. In Case P, for each $\overrightarrow{yx} \in \overrightarrow{G}_{\mathrm{ex}}$ independently let $\mathbb{P}(\overrightarrow{yx} \in J^0_{\mathrm{ex}}) = \frac{7}{8}$ or $\mathbb{P}(\overrightarrow{yx}^- \in J^K_{\mathrm{ex}}) = \frac{1}{8}$.

Some edges of G may not be allocated by this process. Note that arcs in $J[V, W]$ are all directed from V to W, so we will often suppress the direction and think of $J[V, W]$ as a graph.

For $uu' \in F'[A^g_i, A^{g'}_{i'}]$ with $i, i' \in [0, i^*]$ and $g, g' \in \{\mathrm{hi}, \mathrm{lo}, \mathrm{no}\}$ let every $\overrightarrow{G}^{gg'}_{i0} = \overrightarrow{G}^g_{i0}$ and let $\overrightarrow{G}_{uu'} = \overrightarrow{G}^{gg'}_{ii'}$ and $p_{uu'} = p^{gg'}_{ii'}$, recalling that $p^{gg'}_{i0} = p^g_{i0}$ for all g'.

For $g \in A^\Delta \cup \{\mathrm{lo}, \mathrm{no}\}$, $u \in A^g_i$ let $A_u = A^g_i$, $J_u = J^g_i$, $\alpha_u = \alpha^g_i$.

For $\overrightarrow{xw} \in J_u$ we also write $A_{\overrightarrow{xw}} = A_u$, $J_{\overrightarrow{xw}} = J_u$, $\alpha_{\overrightarrow{xw}} = \alpha_u$.

APPROXIMATE DECOMPOSITION

For $i = 1, \ldots, i^*$ apply the following steps.

i. Let $(\mathcal{H}_i, \omega)$ be the weighted hypergraph $\mathcal{H}_i$ with vertex parts $\overrightarrow{G}_i$, J_i and $A_i \times W$, where for each $u \in A_i$ and $\overrightarrow{xw} \in J_u$ such that $\overleftarrow{xy} \in \overrightarrow{G}_{uv}$ for all $y = \phi_w(v)$ with $v \in N_<(u)$ we include an edge labelled "$\phi_w(u)$=x" consisting of uw, $\overrightarrow{xw}$ and all such $\overleftarrow{xy}$, with weight

$$\omega(\text{“}\phi_w(u){=}x\text{”}) := |A_u|^{-1} \prod_{v \in N_<(u)} p^{-1}_{uv}.$$

ii. Define ω' on $\mathcal{H}_i$ by $\omega'(\boldsymbol{e}) = (1 - .5\varepsilon_i)\omega(\boldsymbol{e})/Q(\boldsymbol{e})$ where $Q(\boldsymbol{e})$ is the maximum of 1 and all $\omega(\mathcal{H}_i[\boldsymbol{v}]) := \sum\{\omega(\boldsymbol{e}) : \boldsymbol{v} \in \boldsymbol{e}\}$ with $\boldsymbol{v} \in \boldsymbol{e}$. Let $\mathcal{M}_i$ be a random matching obtained by applying Lemma 2.8 to $(\mathcal{H}_i, \omega')$. For each "$\phi_w(u)$=x" in $\mathcal{M}_i$ extend ϕ_w by setting $\phi_w(u) = x$.

iii. For each $a \in A_i$ in any order, let $W_a = \{w \in W : \phi_w(a) \text{ undefined}\}$, let $V_a \in \binom{V}{|W_a|}$ be uniformly random, and define $\{\phi_w(a)w : w \in W_a\}$=MATCH$(B_a, Z_a)$, where $Z_a = \{\phi_w(b)w : b \in N_<(a)\}$ and $B_a \subseteq V_a \times W_a$ consists of all vw with $v \in N_{J'_i}(w) \setminus \mathrm{Im}\,\phi_w$ and each $\phi_w(b)v$ for $b \in N_<(a)$ an unused edge of G'_i.

To avoid confusion, we emphasise that H_i is a digraph and $\mathcal{H}_i$ is a hypergraph. We sometimes use bold font as above to avoid confusion between $\boldsymbol{v} \in V(\mathcal{H}_i)$ and $v \in V(H_i) = V(G)$. We define 'time' during the algorithm by a parameter t taking values in a set $\mathcal{T}$ with the following elements: 0 is the start, t_a for $a \in V(T)$ is the time (if it exists) at which some $\phi_w(a)$ are defined by choosing a matching M_a, t_{hi} is the end of HIGH DEGREES, t_{int} is the end of INTERVALS, t_{G_0} is after choosing G_0 and J_0, t_{**} is the end of embedding A^{**}, t_0 is the end of EMBED A_0, times t_i and t^+_i for $i \in [i^*]$

are just before and just after we extend the embeddings according to the matching $\mathcal{M}_i$ (so t_1 is the end of DIGRAPH). For any time $t \neq 0$ we let t^- be the time just before t.

We write $\mathbb{P}^t$ and $\mathbb{E}^t$ for conditional probability and expectation given the history of the algorithm up to time t. For $t \in \mathcal{T}$ and $w \in W$ let $A_{t,w}$ be the set of w-embedded vertices at time t. We write A_t if it is independent of w.

We denote the graph remaining after the approximate decomposition by $G'_{\rm ex} = G \setminus \bigcup_{w \in W} \phi_w(F)$.

We complete the T-decomposition of G by the 'exact step' algorithms below: we apply SMALL STARS in Case S, PATHS in Case P, or LARGE STARS in Case L.

SMALL STARS

i. For $x \in V(G)$ let L_x be the set of all uw where u is a leaf of a star in $P_{\rm ex}$ with centre $\phi_w^{-1}(x)$.
ii. Let D be a uniformly random orientation of $G'_{\rm ex}$. While not all $d^+_D(x) = |L_x|$, choose uniformly random x, y, z with $|L_x| > d^+_D(x)$, $|L_y| < d^+_D(y)$, $z \in N^+_D(y) \cap N^-_D(x)$ and reverse $\overrightarrow{yz}$, $\overrightarrow{zx}$.
iii. For each $x \in V(G)$ in arbitrary order, define $\phi_w(u)$ for all $uw \in L_x$ by $M_x = \{\{uw, \phi_w(u)\} : uw \in L_x\}$=MATCH$(F_x, \emptyset)$, where $F_x \subseteq L_x \times N^+_D(x)$ consists of all $\{uw, y\}$ with $uw \in L_x$, $y \in N^+_D(x) \cap N_{J_{\rm ex}}(w) \setminus {\rm Im}\,\phi_w$.

PATHS

i. Call $x \in V(G)$ odd if the parity of $d_{G'_{\rm ex}}(x)$ differs from that of the number of w such that $x = \phi_w(a)$ where a is the end of a bare path in $P_{\rm ex}$. Let X be the set of odd vertices. Let $a_1\ell_1$, $a_2\ell_2$ be the leaf edges in $P_{\rm ex}$, with leaves ℓ_1, ℓ_2. Throughout, let $G_{\rm free} = \{$unused edges$\}$.
ii. Define all $\phi_w(\ell_1)$ by $M_1 = \{\phi_w(\ell_1)w : w \in W\}$=MATCH$(B_1, Z_1)$, where $Z_1 = \{\phi_w(a_1)w\}_{w \in W}$ and $B_1 = \{vw : v \in N_{J_{\rm ex}}(w), v\phi_w(a_1) \in G_{\rm free}\}$.
iii. Fix $X' \subseteq X$, $W' \subseteq W$ with $|X'| = |W'| = |X|/2$. Define $\phi_w(\ell_2)$ for $w \in W'$ by $M'_2 = \{\phi_w(\ell_2)w : w \in W'\}$=MATCH$(B'_2, Z'_2)$, where $Z'_2 = \{\phi_w(a_2)w\}_{w \in W'}$ and $B'_2 = \{vw : w \in W', v \in N_{J_{\rm ex}}(w) \cap X', v\phi_w(a_2) \in G_{\rm free}\}$.
iv. Let $V' = (V \setminus X) \cup X'$. Define $\phi_w(\ell_2)$ for $w \in W \setminus W'$ by $M_2 = \{\phi_w(\ell_2)w : w \in W \setminus W'\}$=MATCH$(B_2, Z_2)$, where $Z_2 = \{\phi_w(a_2)w\}_{w \in W \setminus W'}$ and $B_2 = \{vw : w \in W \setminus W', v \in N_{J_{\rm ex}}(w) \cap V', v\phi_w(a_2) \in G_{\rm free}\}$.
v. For each $w \in W$ fix $8d(x,y)$-paths P_w^{xy} for each $[x,y] \in \mathcal{Y}_w$ centred in vertex-disjoint bare $(8d(x,y)+2)$-paths in $P_{\rm ex}$. Extend each ϕ_w to an embedding of $P_{\rm ex} \setminus \bigcup_{xy} P_w^{xy}$ so that $\phi_w^{-1}(x)$, $\phi_w^{-1}(y^+)$ are the ends of P_w^{xy}, according to a random greedy algorithm, where in each step, in any order, we define some $\phi_w(a) = z$, uniformly at random with $z \in J_{\rm ex}(w) \setminus {\rm Im}\,\phi_w$ and $zz' \in G_{\rm free}$ whenever $z' = \phi_w(b)$ with $b \in N_T(a)$.
vi. Apply Theorem 4.6 to decompose $G_{\rm free}$ into $(G_w : w \in W)$ such that each G_w is a vertex-disjoint union of $8d(x,y)$-paths $\phi_w(P_w^{xy})$, $[x,y] \in \mathcal{Y}_w$ internally disjoint from ${\rm Im}\,\phi_w$.

LARGE STARS

i. Let $\mathcal{S}$ be the union of all maximal leaf stars in T that have size $\geq\Lambda$. Let $F = T \setminus \mathcal{S}$.
Let S be the set of star centres of $\mathcal{S}$ and $S^+ = \{v \in V(T) : d_T(v) \geq \Lambda\}$.
Partition W as $W_1 \cup W_2 \cup W_3$ with each $||W_i| - n/3| < 1$. For each $v \in V(G)$ independently choose exactly one of $\mathbb{P}(v \in U_i^a) = d_{\mathcal{S}}(a)/3|\mathcal{S}|$ with $a \in S$, $i \in [3]$. Let $U_i = \bigcup_a U_i^a$.
While $\sum_{i=1}^3 ||W_i| - |U_i|| > 0$ relocate a vertex between the U_i^a so as to decrease this sum.
ii. Fix an order $\prec$ on $V(F)$ starting with some $u_0 \in S^+$ such that $N_<(u) = \{v \prec u : vu \in F\} = \{u^-\}$ has size 1 for all $u \neq u_0 \in V(F)$. Fix distinct $\phi_w(u_0)$, $w \in W$ with $\phi_w(u_0) \in U_i$ whenever $w \in W_i$.

iii. Throughout, update $G_{\text{free}} = \{\text{unused edges}\}$, the image $\operatorname{Im}\phi_w$ of ϕ_w, and a digraph J on $V(G)$ consisting of all $\overrightarrow{yx}$ with $y = \phi_w(a)$ and $x \in U^a \cap \operatorname{Im}\phi_w$ for some $w \in W$, $a \in S$.

iv. For each $a \in V(F) \setminus \{u_0\}$ in $\prec$ order let $M_i^a = \{\phi_w(a)w : w \in W_i\}$=MATCH$(B_i^a, Z_i^a)$, $i \in [3]$, thus defining all $\phi_w(a)$, with B_i^a, Z_i^a as follows.

- If $a \notin S^+$ let $Z_i^a = \{\phi_w(a^-)w\}_{w \in W_i}$ and define $B_i^a \subseteq U_{i-1} \times W_i$ (with $U_0 := U_3$) by
$$N_{B_i^a}(w) = A_a^w := U_{i-1} \cap N_{G_{\text{free}}}(\phi_w(a^-)) \setminus \operatorname{Im}\phi_w.$$
If $|U_{i-1}| < |W_i|$, choose $\phi_w(a)w \in B_i^a$ uniformly at random, update B_i^a and remove w from its vertex set. If $|U_{i-1}| > |W_i|$, remove some randomly chosen $u \in U_{i-1}$ from B_i^a.
- If $a \in S^+$ let $Z_i^a = \big\{vw : v \in \{\phi_w(a^-)\} \cup (U^a \cap \operatorname{Im}\phi_w)\big\}_{w \in W_i}$ and define $B_i^a \subseteq U_i \times W_i$ by $N_{B_i^a}(w) = A_a^w = \bigcup_{b \in S} A_a^{wb}$ where
$$A_a^{wb} = U_i^b \cap N_{G_{\text{free}}}(\phi_w(a^-)) \setminus \big(\operatorname{Im}\phi_w \cup N_J^+(\operatorname{Im}\phi_w \cap U^a) \cup N_J^-(\phi_w(b))\big).$$

v. Orient G_{free} as $D = \bigcup_{w \in W} D_w$, where for each $xy \in G_{\text{free}}$ with $x \in U^a$ and $y \in U^b$, if $\overrightarrow{xy} \in J$ we have $\overrightarrow{yx} \in D_w$ where $\phi_w(a) = y$, if $\overrightarrow{yx} \in J$ we have $\overrightarrow{xy} \in D_w$ where $\phi_w(b) = x$, or otherwise we make one of these choices independently with probability 1/2.

vi. While $\Sigma := \sum_{w \in W} \sum_{a \in S} |d^+_{D_w}(\phi_w(a)) - d_{\mathcal{S}}(a)| > 0$, we fix $u = \phi_w(a)$ with $d^+_{D_w}(u) < d_{\mathcal{S}}(a)$ and $u' = \phi_{w'}(a')$ with $d^+_{D_{w'}}(u') < d_{\mathcal{S}}(a')$, and apply a uniformly random *xvz-move* for $uwu'w'$, defined as follows. Choose xvz with $\{\overleftarrow{vu}', \overleftarrow{zu}', \overrightarrow{vx}, \overrightarrow{xu}\} \subseteq D$ unmoved, with $x \notin \operatorname{Im}\phi_w$, with $v \notin \operatorname{Im}\phi_{w'} \cup \operatorname{Im}\phi_{w^x}$ where $\phi_{w^x}(b) = x$, $u \in U^b$, with $u' \notin \operatorname{Im}\phi_{w^v}$ where $\phi_{w^v}(c) = v$, $x \in U^c$, and with $z \in N^+_{D_{w'}}(u') \setminus \operatorname{Im}\phi_{w^{u'}}$ where $\phi_{w^{u'}}(d) = u'$, $v \in U^d$. The xvz-move for $uwu'w'$ reverses the path $u'vxu$ in D, assigning $\overrightarrow{ux} \in D_w$, $\overrightarrow{xv} \in D_{w^x}$, $\overrightarrow{vu}' \in D_{w^v}$ and $\overleftarrow{zu}' \in D_{w^{u'}}$.

2.2 Preliminaries

Here we gather some well-known results concerning concentration of probability and Szemerédi regularity, and also a result on random perfect matchings in quasirandom bipartite graphs, which is perhaps new (although the proof technique via switchings is somewhat standard).

We start with the following classical inequality of Bernstein (see e.g. [4, (2.10)]) on sums of bounded independent random variables. (In the special case of a sum of independent indicator variables we will simply refer to the 'Chernoff bound'.)

Lemma 2.1. *Let $X = \sum_{i=1}^n X_i$ be a sum of independent random variables with each $|X_i| < b$. Let $v = \sum_{i=1}^n \mathbb{E}(X_i^2)$. Then $\mathbb{P}(|X - \mathbb{E}X| > t) < 2e^{-t^2/2(v+bt/3)}$.*

We also use McDiarmid's bounded differences inequality, which follows from Azuma's martingale inequality (see [4, Theorem 6.2]).

Definition 2.2. Suppose $f : S \to \mathbb{R}$ where $S = \prod_{i=1}^n S_i$ and $b = (b_1, \dots, b_n) \in \mathbb{R}^n$. We say that f is *b-Lipschitz* if for any $s, s' \in S$ that differ only in the ith coordinate we have $|f(s) - f(s')| \le b_i$. We also say that f is *v-varying* where $v = \sum_{i=1}^n b_i^2/4$.

Lemma 2.3. *Suppose $Z = (Z_1, \dots, Z_n)$ is a sequence of independent random variables, and $X = f(Z)$, where f is v-varying. Then $\mathbb{P}(|X - \mathbb{E}X| > t) \le 2e^{-t^2/2v}$.*

We say that a random variable is *(μ, C)-dominated* if we can write $Y = \sum_{i \in [m]} Y_i$ such that $|Y_i| \le C$ for all i and $\sum_{i \in [m]} \mathbb{E}'|Y_i| < \mu$, where $\mathbb{E}'|Y_i|$ denotes the expectation conditional on any given values of Y_j for $j < i$. The following lemma follows easily from Freedman's inequality [11].

Lemma 2.4. *If Y is (μ, C)-dominated, then $\mathbb{P}(|Y| > 2\mu) < 2e^{-\mu/6C}$.*

Next we recall some definitions (not quite in standard form) pertaining to Szemerédi regularity. A bipartite graph $B \subseteq X \times Y$ with $|B| = d|X||Y|$ is *ε-regular* if $|B[X', Y']| = d|X'||Y'| \pm \varepsilon|X||Y|$ for every $X' \subseteq X$, $Y' \subseteq Y$. If also $|B(x) \cap Y| = (1 \pm \varepsilon)d|Y|$ and $|B(y) \cap X| = (1 \pm \varepsilon)d|X|$ for all $x \in X$, $y \in Y$ then B is *ε-super-regular.* We will need the well-known 'pair condition' discovered independently by several pioneers in the theory of Szemerédi regularity (we refer to [23] for the history and a version of the following statement).

Lemma 2.5. *Let $\varepsilon < 2^{-200}$ and $B \subseteq X \times Y$ with $|X| = |Y| = m$, where $|N_B(x) \cap Y| > (d - \varepsilon)m$ for all $x \in X$ and $|N_B(xx') \cap Y) < (d + \varepsilon)^2 m$ for all but $\leq 2\varepsilon m^2$ pairs xx' in X. Then B is $\varepsilon^{1/6}$-regular.*

We also require the following lemma; the proof is standard, so we omit it.

Lemma 2.6. *Let $n^{-1} \ll \alpha \ll \beta \ll d, r^{-1}, D^{-1}$ and G be an α-super-regular bipartite graph with parts X and Y of size $\geq n$ and density $d(G) \geq d$. Suppose H is a $\leq$r-multigraph on Y of maximum degree D. Then for all but at most $\beta|X|$ vertices x we have $\sum_{e \in H[N_G(x)]} d(G)^{-|e|} = |H| \pm \beta|Y|$.*

Next we present a result on random perfect matchings in super-regular bipartite graphs. Given $M, Z \subseteq X \times Y$, an $MZMZ$ is a 4-cycle that alternates between M and Z. We also write $MZMZ$ for the number of $MZMZ$'s.

Lemma 2.7. *Let $n^{-1} \ll \alpha \ll d$ and $B, Z \subseteq X \times Y$ with $|X| = |Y| = n$. Suppose Z has maximum degree $< n^{.4}$ and B is α-super-regular with density $d(B) \geq d$. Then there is a distribution on perfect matchings M of B with $MZMZ = 0$ such that $\mathbb{P}(xy \in M) = (1 \pm \alpha^{.98})(d(B)n)^{-1}$ for any edge xy, and for any $X' \subseteq X$, $Y' \subseteq Y$ whp $|M[X', Y']| = |B[X', Y']|(d(B)n)^{-1} \pm n^{.8}$.*

Proof. Let $\mathcal{M}$ be the set of perfect matchings of B. It is well-known (and easy to see by Hall's theorem) that $\mathcal{M} \neq \emptyset$. We consider a Markov chain on $\mathcal{M}$ where the transition from any $M \in \mathcal{M}$ is a uniformly random swap, defined by choosing a 6-cycle C in B that is M-alternating (every other edge is in M) and swapping $C \cap M$ with $C \setminus M$, subject to the new edges $C \setminus M$ not forming any new $MZMZ$'s. It is well-known that every Markov chain on a finite state space has a stationary distribution (which is not necessarily unique). Fix some stationary distribution μ and let $M \sim \mu$.

To analyse the chain, we start with an estimate for the number of swaps for any given M. Let G_M be the auxiliary tripartite graph with parts X_1, X_2, X_3 each a copy of X, where for $i \in [3]$, $x_i \in X_i$, $x'_{i+1} \in X_{i+1}$ we have $x_i x'_{i+1} \in G_M$ if $M(x_i)x'_{i+1} \in B \setminus M$ (and $X_4 := X_1$). Note that M-alternating 6-cycles in G correspond to triangles in G_M. Each $G_M[X_i, X_{i+1}]$ is a copy of $B \setminus M$, so is 2α-super-regular, and so by the triangle counting lemma G_M has $(1 \pm \alpha^{.99})(d(B)n)^3$ triangles. Each edge in M forms an $MZMZ$ with $\leq n^{.8}$ other edges, each forbidding $\leq n$ possible swaps, so the number of swaps is $(1 \pm \alpha^{.99})(d(B)n)^3 \pm n^{2.8} = (1 \pm 1.1\alpha^{.99})(d(B)n)^3$.

Next we claim that μ is supported on $\mathcal{M}_0 := \{M : MZMZ = 0\}$. To see this, first note that in any step of the chain $MZMZ$ is non-increasing. Also, the M-alternating 6-cycles that remove any given e from M correspond to triangles in G_M containing some given vertex. There are $(1 \pm \alpha^{.99})d(B)^3 n^2$ such triangles, of which $\leq n^{1.8}$ are forbidden. Letting p_M^{-e} denote the probability that e is removed by a transition from M we have $p_M^{-e} = (1 \pm 2.2\alpha^{.99})n^{-1}$. In particular, if $MZMZ > 0$ then it decreases with positive probability. Thus $\mathcal{M}_0$ is an absorbing class, so the claim holds.

Next we estimate $\mathbb{P}(e \in M)$ for any given $e \in B$. Let $\mathcal{M}[e] = \{M \in \mathcal{M} : e \in M\}$. For $M \in \mathcal{M} \setminus \mathcal{M}[e]$ let p_M^{+e} denote the probability that e is added by a transition from M. The M-alternating 6-cycles for adding e correspond to a choice in some common neighbourhood $N_{G_M}(x_1) \cap$

$N_{G_M}(x_2)$. Thus there are $(1 \pm \alpha^{.99})d(B)^2 n$ such 6-cycles, of which $\le 4n^{.4}$ are forbidden, so $p_M^{+e} = (1 \pm 2.2\alpha^{.99})d(B)^{-1}n^{-2}$. Now

$$\mathbb{P}(e \in M) = \sum_{M \in \mathcal{M}[e]} \mu_M = \sum_{M \in \mathcal{M}[e]} \mu_M (1 - p_M^{-e}) + \sum_{M \in \mathcal{M} \setminus \mathcal{M}[e]} \mu_M p_M^{+e},$$

so $\sum_{M \in \mathcal{M}[e]} \mu_M p_M^{-e} = \sum_{M \in \mathcal{M} \setminus \mathcal{M}[e]} \mu_M p_M^{+e}$ and so $(1 \pm 4.5\alpha^{.99})n^{-1}\mathbb{P}(e \in M) = d(B)^{-1}n^{-2}\mathbb{P}(e \notin M)$, giving $\mathbb{P}(e \in M) = (1 \pm 5\alpha^{.99})(d(B)n)^{-1}$.

To obtain the final property, we consider uniformly random partitions $(X_i : i \in I)$ of X and $(Y_i : i \in I)$ of Y with each $|X_i| = |Y_i| = \sqrt{n} \pm 1$. We let $M = \bigcup_{i \in I} M_i$ where each $M_i \sim \mu_i$ independently with μ_i a stationary distribution of the above chain for $B_i = B[X_i, Y_i]$ and $Z_i = Z[X_i, Y_i]$. By Chernoff bounds whp each B_i is 1.1α-super-regular with $d(B_i) = d(B) \pm n^{-.1}$. By the above analysis, each $\mathbb{P}(e \in M) = \sum_i \mathbb{P}(e \in M_i) = \sum_i (n^{-1}|X_i|)^2 (1 \pm 5(1.1\alpha)^{.99})(d(B_i)|X_i|)^{-1} = (1 \pm 6\alpha^{.99})(d(B)n)^{-1}$.

It remains to estimate $|M[X', Y']| = \sum_i M[X_i', Y_i']$, where $X_i' = X' \cap X_i$, $Y_i' = Y' \cap Y_i$. By Chernoff bounds, whp each $B[X_i', Y_i'] = n^{-1}|B[X', Y']| \pm n^{.76}$, so

$$\mathbb{E}|M[X_i', Y_i']| = (1 \pm 6\alpha^{.99})(d(B)\sqrt{n})^{-1}(n^{-1}|B[X', Y']| \pm n^{.76}).$$

Also, $\mathbb{E}|M[X_i', Y_i']|^2 = \mathbb{E}|M[X_i', Y_i']| + \sum_{e \ne e' \in B[X_i', Y_i']} \mathbb{P}(\{e, e'\} \subseteq M_i) < 2n$, as each $\mathbb{P}(\{e, e'\} \subseteq M_i) = (1 \pm 6\alpha^{.99})(d(B_i)|X_i|)^{-2}$ by similar arguments to those above. The required estimate for $|M[X', Y']|$ now follows from Lemma 2.1. □

We conclude this subsection with a result on matchings in weighted hypergraphs, along the lines of the literature stemming from the Rödl nibble mentioned in the overview above. The following lemma is a slight adaptation of a convenient general setting of the nibble recently provided by Ehard, Glock and Joos [8]. Given a weighted hypergraph (H, ω), we call a function $f : \binom{H}{\le r} \to \mathbb{R}$ clean if $f(I) = 0$ whenever I is not a matching. For $H' \subseteq H$ let $f(H') = \sum\{f(E) : E \in \binom{H'}{\le r}\}$, and $f(H', \omega) = \sum\{\omega(E)f(E) : E \in \binom{H'}{\le r}\}$, where $\omega(E) = \prod_{e \in E} \omega(e)$. For $S, T \in \binom{H}{\le r}$ we also let $f_S(T) = f(S \cup T)$ if $T \cap S = \emptyset$, or $f_S(T) = 0$ otherwise.

Lemma 2.8. *Let $C^{-1} \ll \alpha \ll \beta \ll r^{-1}, \ell^{-1}$ and (H, ω) be a weighted $\le r$-graph with $\omega(e) \ge C^{-1}$ for all $e \in H$, and $\omega(H[v]) \le 1$, $\omega(H[uv]) < C^{-\beta}$ for all $u \ne v \in V(H)$. Suppose f is a clean function on $\binom{H}{\le \ell}$ with $f_S(H, \omega) \le C^{-\beta} f(H, \omega)$ whenever $S \ne \emptyset$. Then there is a distribution on matchings M in H such that $f(M) = (1 \pm C^{-\beta})f(H, \omega)$ with probability $\ge 1 - e^{-C^{\alpha}}$.*

The proof of Lemma 2.8 is essentially the same as that of [8, Theorem 1.3], with a few modifications as follows. The statement in [8]:

- applies to unweighted hypergraphs of maximum degree Δ and maximum codegree $< \Delta^{1-\beta}$; our version can be reduced to this version by considering a multihypergraph where the multiplicity of an edge e is $\lfloor \Delta^2 \omega(e) \rfloor$, say.
- gives a (deterministic) matching M satisfying the required conclusion for a suitably small set of functions f; this is obtained by proving the existence of a distribution on matchings as in our statement and taking a union bound,
- applies to functions on $\binom{H}{\ell}$, from which a version for functions on $\binom{H}{\le \ell}$ is easily deduced.

2.3 Tree partition

We start our analysis of the algorithm by considering the subroutine TREE PARTITION.

Lemma 2.9. *We can choose a case in {L,S,P} for T and we have $i^* < i^+$. Also, $|A_0| \le 6\varepsilon n$, each $|A_i^a| \ge \Delta$ and $|A_i^{\circ_j}| \ge \delta n$ if non-empty, with $A_i^a \cap A_{i'}^{a'} = \emptyset$ for $ai \ne a'i'$.*

Proof. To see that we can choose a case for T, we suppose that T does not satisfy Case L or Case S, and show that it must satisfy Case P. Here we rely on the well-known fact that any tree with few leaves must have many vertices in long bare paths (we will use the precise statement given by [27, Lemma 4.1]). Let T' be the tree obtained from T by removing all leaf stars of size $\ge\Lambda$. Then $|V(T')| \ge p_+ n$, as T does not satisfy Case L.

We claim that T' has $< 2p_- n$ leaves. To see this, let $\mathcal{S}$ be the set of maximal leaf stars of T'. For each $S \in \mathcal{S}$ obtain S' from S by deleting all leaves of T' that are not leaves of T. Note that $|S'| \le \Lambda$, or we would have removed S when defining T'. Then $\sum_{S\in\mathcal{S}} |S'| < p_- n$, as T does not satisfy Case S. Also, $\sum_{S\in\mathcal{S}} |S \setminus S'| < n/\Lambda$ as each leaf in any $S \setminus S'$ is the centre of a leaf star in T of size $\ge\Lambda$. The claim follows.

Now [27, Lemma 4.1] implies that T' has $> p_+ n/50K$ vertex-disjoint bare $8K$-paths. At most n/Λ of these contain the centre of some star removed when obtaining T' from T, so $> p_+ n/100K$ are bare paths of T, as required.

Next we bound i^*. Recall that at step $i \ge 1$ we let $B_i = V(F^*) \setminus \bigcup_{j<i} C_j$ and C'_i be the set of $v \in B_i$ with $d_{F^*[B_i]}(v) \le 3$ and $d_{F^*[\bigcup_{j<i} C_j]}(v) \le p_{\max}^{-1}$. We have $|C'_i| > |B_i|/3 - 2p_{\max} n$, as $< 2p_{\max} n$ vertices fail the second condition, and the set X of vertices failing the first condition satisfies $3|X| \le \sum_{x\in X} d_{F^*[B_i]}(v) < 2|B_i|$. Next we let C_i be a maximum independent set in $F^*[C'_i]$; we have $|C_i| \ge |C'_i|/2$ as trees are bipartite. If $|C_i| < \varepsilon n$ we let $i^* = i - 1$ and stop, otherwise we proceed to the next step, noting that $|C_i| > |B_i|/7$. There can be at most $i^+ = 7\log\varepsilon^{-1}$ steps, otherwise we would continue past a step i with $|B_i| < (6/7)^{i^+} n < \varepsilon n$.

For the remaining statements, we first note that the bounds for and disjointness of the sets A_i^a are immediate from the algorithm and the definition of A_0 as a span. Finally we consider step (iv) of TREE PARTITION. For each $j \in [4]$, there are at most i^+ steps where we move some $A_i^{\circ_j}$ to A_0 if it has size $< \delta_j n$, thus adding $< 5\delta_j n$ vertices to A_0 after including any forced by the definition as a span. Note that by choice of the order $\circ_1, \dots, \circ_4$ it is not possible for some $A_i^{\circ_j}$ to be moved to A_0 and then to reappear at a later step. At the end of the process, any surviving $A_i^{\circ_j}$ has size $|A_i^{\circ_j}| > \delta_j n - \sum_{j'>j} 5 i^+ \delta_{j'} n \ge \delta_j n(1 - 5(4-j)\delta^{-.1} i_+) \ge \delta n$. This completes the proof. □

2.4 High degrees

Continuing through the algorithm, the following lemma shows that the subroutine HIGH DEGREES is whp successful, and the image of each embedding is well-distributed with respect to common neighbourhoods in G.

Lemma 2.10. *whp HIGH DEGREES does not abort, and $\mathbb{P}^{t_a^-}(\phi_w(a) \in N_G(S)) = (1\pm\delta)p^{|S|}$ for any $a \in A^*$, $w \in W$ and $S \subseteq V(G)$, $|S| \le s$.*

We make some preliminary observations before giving the proof. First, we write the proof assuming $|N_<(a)| \le 4$ for all $a \in A^*$ rather than using our real bound $|N_<(a)| \le 1$, so that it is more obvious how to apply the same proof to obtain Lemma 2.11. We note that the choices of x_a for $a \in A^*$ are possible. Indeed, at each step, we forbid $\le 6d|A^*| \le 30dn/\Delta$ choices of x_a with $d(x_a, x_{a'}) \le 3d$ for some $a' \prec a$, and $< 5n/\Delta$ choices with $d(x_a, x_{a'}) = d(x_b, x_{b'})$ for some $a' \in N_<(a)$ and $bb' \in F[<a]$. We also note the following estimate for common neighbourhoods, which is immediate from a (hypergeometric) Chernoff bound: whp for any $S \subseteq V(G)$ with $|S| \le s$ and $X = V_0$ or $X = V_{v^*}$ with $v^* \in V^*$ we have $|N_G(S) \cap X| = ((1 \pm 1.1\xi)p)^{|S|}|X|$.

Proof. We can condition on partitions of V and W satisfying the above estimates for $|N_G(S) \cap X|$. First we consider the choices of $(M_a^* : a \in A^*)$, which are independent of $(M_a^0 : a \in A^*)$. For each $a \in A^*$, $w^* \in W^*$, $v^* = x_a + w^*$ we will show that Lemma 2.7 applies to choose $M_a^*[V_{v^*}, W_{w^*}]$=MATCH$(B_a^{w^*}, Z_a^{w^*})$, where $B_a^{w^*} \subseteq V_{v^*} \times W_{w^*}$ satisfies $B_a^{w^*}(w) = N_{G_{t_a^-}}(\phi_w(N_<(a))) \cap V_{v^*} \setminus \phi_w(<a)$, where $G_t \subseteq G$ is the graph of unused edges at time t. For each $v \in V$ there is a unique edge $w\phi_w(a) \in M_a$ with $\phi_w(a) = v$, which uses $|N_<(a)|$ edges at v, so $G \setminus G_t$ has maximum degree $\le |T[A^*]| \le 5n/\Delta$. Note that the constraint that $\phi_w(a)\phi_w(a')$ is unused for all $a' \in N_<(a)$ is automatically satisfied as $d(x_a, x_{a'}) \neq d(x_b, x_{b'})$ for all $bb' \in F[<a]$.

We also note that $Z_a = \{\phi_w(b)w : b \in N_<(a)\}$ has maximum degree ≤ 4.

At time t, let $H^t_{w^*}$ be the hypergraph on $V(G)$ with edges $e^t_w = \phi_w(N_<(a) \cap A_t)$ for $w \in W_{w^*}$. Note that $H^t_{w^*}$ is a matching, as M_b^* is a matching for each $b \in N_<(a)$ and $\phi_w(b) \in V_{w^*+x_b}$ with distinct x_b. We let $\mathcal{B}_t$ be the 'bad' event that $|H^t_{w^*}[N_G(v)]| \neq ((1 \pm 7\xi)p)^{|N_<(a) \cap A_t|} n_*$ for some $v \in V(G)$, $a \in A^*$, $w^* \in W^*$. We let τ be the smallest t such that $\mathcal{B}_t$ occurs, or ∞ if there is no such t. We fix $a \in A^*$ and bound $\mathbb{P}(\tau = t_a)$.

We claim that $B_a^{w^*}$ is ξ'-super-regular of density $(1 \pm 5\xi)p^{|N_<(a)|}$. To show this, we first lighten our notation, writing $t = t_a^-$ and $H = H^t_{w^*}$, which has edges $e_w = \phi_w(N_<(a))$ for $w \in W_{w^*}$. Any $v \in V_{v^*}$ has degree $|H[N_{G_t}(v)]| \pm |\{w \in W_{w^*} : v \in \phi_w(<a)\}| = |H[N_G(v)]|$ as $v\phi_w(a')$ is unused for all $a' \in N_<(a)$ and all $w \in W_{w^*}$, and $v = \phi_w(a'')$ for some $a'' \in <a$ and $w \in W_{w^*}$ iff $x_a = v^* - w^* = x_{a''}$. As $\mathcal{B}_t$ does not hold for $t < t_a$, v has degree $((1 \pm 7\xi)p)^{|N_<(a)|} n_*$. Any $w \in W_{w^*}$ has degree $|N_G(\phi_w(N_<(a))) \cap V_{v^*}| = ((1 \pm 1.1\xi)p)^{|N_<(a)|} n_* = (1 \pm 5\xi)p^{|N_<(a)|} n_*$. For any $V' \subseteq V_{v^*}$, $W' \subseteq W_{w^*}$ we have $|B_a^{w^*}[V', W']| = \sum_{w \in W'} |N_G(\phi_w(N_<(a)) \cap V'| = \sum_{v \in V'} |H'[N_G(v)]|$, where $H' = \{e_w : w \in W'\}$, so $|B_a^{w^*}[V', W']| = p^{|N_<(a)|}|V'||W'| \pm \xi' n_*^2$ by Lemma 2.6. This proves the claim.

Thus Lemma 2.7 applies, giving $\mathbb{P}^{t_a^-}(vw \in M_a^*) = (1 \pm \delta/2)(p^{|N_<(a)|} n_*)^{-1}$ for any $vw \in B_a^{w^*}$, so $\mathbb{P}^{t_a^-}(\phi_w(a) \in N_G(S)) = (1 \pm \delta)p^{|S|}$ for any $w \in W_{w^*}$ and $S \subseteq V(G)$, $|S| \le s$.

To bound $\mathbb{P}(\mathcal{B}_{t^a})$, note that $H^t_{w^*}$ only changes when we choose M_b^* for $b \in N_<(a)$. Fix v and write $W^t = \{w : e^t_w \in H^t_{w^*}[N_G(v)]\}$, where $t = t_b^-$. For any $w \in W^t$ we have $e^{t_b}_w \in H^{t_b}_{w^*}$ iff $\phi_w(b) \in N_G(v)$, so $|H^{t_b}_{w^*}| = |M_b^*[W^t, N_G(v)]|$, which by Lemma 2.7 is whp $|B_a^{w^*}[N_G(v), W^t]|((1 \pm 5\xi)p^{|N_<(a)|} n)^{-1} \pm n^{.8} = (1 \pm 7\xi)p|W^t|$. Thus whp $\mathcal{B}_{t^a}$ does not hold for any a, so $\tau = \infty$.

Now we consider the choice of M_a^0=MATCH(B_a^0, Z_a^0), where $Z_a = Z_a[V_0, W_0]$ and $B_a \subseteq V_0 \times W_0$ is defined by $B_a^0(w) = N_{G_t}(\phi_w(N_<(a))) \cap V_0 \setminus \phi_w(<a)$. Let H^t_a be the hypergraph on V_0 with edges $e^t_w = \phi_w(N_<(a) \cap A_t) \cap V_0$ for $w \in W_0$. Let $\mathcal{B}'_t$ be the 'bad' event that for some $a \in A^*$ we have some $|H^t_a[N_{G_t}(v)]| \neq ((1 \pm \xi')p)^{|N_<(a) \cap A_t|} n_0$. We let τ' be the smallest t such that $\mathcal{B}'_t$ occurs, or ∞ if there is no such t. We fix $a \in A^*$ and bound $\mathbb{P}(\tau' = t_a)$.

Similarly to the arguments for M_a^*, as $\mathcal{B}'_t$ does not hold, B_a is ξ'-super-regular of density $(1 \pm 5\xi)p^{|N_<(a)|}$, using the maximum degree bound on $G \setminus G_t$ to estimate common neighbourhoods $|N_{G_t}(S) \cap V_0|$ for degrees of $w \in W_0$, recalling that $n_0 > n\Delta^{-.1}/2$, and estimating $|B_a[V', W']| = \sum_{v \in V'} |H'[N_{G_t}(v)]|$ by Lemma 2.6 applied to G_t and $H' = \{e^t_w : w \in W'\}$, which has maximum degree at most 16, as for each $b, b' \in N_<(a)$ and $w \in W_0$ there is a unique $w' \in W_0$ with $\phi_w(b) = \phi_{w'}(b')$.

Again we similarly deduce $\mathbb{P}^{t_a^-}(vw \in B_a) = (1 \pm \delta/2)(p^{|N_<(a)|} n_0)^{-1}$ for any $vw \in B_a$, so $\mathbb{P}^{t_a^-}(\phi_w(a) \in N_G(S)) = (1 \pm \delta)p^{|S|}$ for any $w \in W_0$ and $S \subseteq V(G)$, $|S| \le s$. To bound $\mathbb{P}(\mathcal{B}'_{t^a})$, note that the same argument as for M_a^* gives whp $|H^t_a[N_G(v)]| = ((1 \pm \xi'/2)p)^{|N_<(a) \cap A_t|} n_0$, and by the maximum degree bound on $G \setminus G_t$ we can replace G by G_t in this estimate, changing $\xi'/2$ to ξ'. Thus whp $\mathcal{B}'_{t^a}$ does not hold for any a, so $\tau' = \infty$, as required. □

Note that $G \setminus G^*$ has maximum degree $\le |T[A^*]| < 5n/\Delta$, as for each $v \in V(G)$ there is a unique edge $w\phi_w(a) \in M_a$ with $\phi_w(a) = v$, which uses $|N_<(a)|$ edges at v. Thus G^* is $(1.1\xi, s)$-typical, so whp the graphs G_0 and G_1 defined in EMBED A_0 are $(1.2\xi, s)$-typical.

We omit the proof of the following lemma, as it is similar to and simpler than the previous.

Lemma 2.11. *For any $a \in A_0 \setminus A^*$, $w \in W$, $x, y \in V(G)$, writing A_a^w for the set of y such that $\phi_w(a) = y$ is possible given the history at time t_a^-, whp $\mathbb{P}^{t_a^-}(\phi_w(a) = y) = (1 \pm D^{-.9} \pm \alpha_0^{.9} 1_{a \in A'_0})|A_a^w|^{-1}$, so whp every $\mathbb{P}^{t_a^-}(\phi_w(a) \in N_{G_1}(x)) = (1 \pm D^{-.9} \pm \alpha_0^{.9} 1_{a \in A'_0})p_1$.*

2.5 Intervals

Next we record some properties of the subroutine INTERVALS that are needed for the exact step in Case P (handled by the subroutine PATHS). We omit the proof, which is essentially the same as that of the corresponding lemma in [20] (the only change is the deletion of the negligible sets $\phi_w(A^*)$). We say that $S \subseteq [n]$ is *d-separated* if $d(a, a') \ge d$ for all distinct a, a' in S. For disjoint $S, S' \subseteq [n]$ we say (S, S') is *d-separated* if $d(a, a') \ge d$ for all $a \in S$, $a' \in S'$.

Lemma 2.12. *In Case P,*

i. $\mathbb{P}^{t_{\text{hi}}}(x \in \overline{X_w}) = \overline{p}_w \pm \Delta^{-.9}$ *for all $w \in W$ and $x \in V(G)$,*

ii. *any subset of $\{\{x \in \overline{X_w}\} : w \in W, x \in V(G)\}$ is independent if it does not include any pair $\{x \in \overline{X_w}\}$, $\{x' \in \overline{X_w}\}$ with $d(x, x') \le 3d$,*

iii. *whp $|\mathcal{Y}(I)| = t_i = \frac{(1-\eta)|P_{\text{ex}}|}{8(2s+1)d_i} \pm n\Delta^{-.9}$ for all $I \in \mathcal{I}^i$, $i \in [2s+1]$,*

iv. *whp all $|Y_w| = (1-\eta)|P_{\text{ex}}|/8 \pm n\Delta^{-.9}$,*

v. *for any $U \subseteq V(G)$, whp for any disjoint $R, R' \subseteq W$ of sizes $\le s$ we have*

$$\left| U \cap N^-_{J_{\text{int}}}(R) \cap N^-_{\overline{J}}(R') \right| = |U| (\tfrac{1}{8}(1-\eta)p_{\text{ex}})^{|R|} \prod_{w \in R'} \overline{p}_w \pm n\Delta^{-.1}$$

where $J_{\text{int}} = \{\overrightarrow{xw} : x \in Y_w\}$ and $\overline{J} = \{\overrightarrow{xw} : x \in \overline{X}_w\}$,

vi. *whp for any disjoint $S, S' \subseteq V$ of sizes $\le s$,*

$$\text{If } S \cup S' \text{ is } 3d\text{-separated then } |\{w : S \subseteq Y_w, S' \subseteq \overline{X_w}\}| = \sum_{w \in W} (\tfrac{1}{8}(1-\eta)p_{\text{ex}})^{|S|} \overline{p}_w^{|S'|} \pm n\Delta^{-.1},$$

$$\text{If } (S, S') \text{ is } 3d\text{-separated then } |\{w : S \subseteq Y_w, S' \subseteq \overline{X_w}\}| \ge 2^{-2s} n (\tfrac{1}{8}(1-\eta)p_{\text{ex}})^{|S|}.$$

2.6 Digraph

Our next lemma summarises various properties of the decompositions of G and $W \times V(G)$ constructed in the subroutine DIGRAPH. Many of these properties are straightforward consequences of the definition and Chernoff bounds. The most significant conclusion is part (viii), showing that the high degree digraph H allocates roughly the correct number of edges to each vertex x for each role ai where $i \in [i^*]$, $a \in A_i^\Delta$. For each such ai we let M'_{ai} consist of all v^*w^* with some label ℓ_{aij}, where $v^*w^* \in M'_\Lambda$ if $(a, i) \in Q^\Lambda$ or $v^*w^*\ell_{aij} \in \mathcal{M}$ if $(a, i) \in Q^\Delta$.

We write G_{ex} for the underlying graph of $\overrightarrow{G}_{\text{ex}}$ and define other underlying graphs similarly. We define $J_{\text{ex}}^{K'}$ by $J_{\text{ex}}^{K'}[V, W] = J_{\text{int}} = \{\overrightarrow{xw} : x \in Y_w\}$ and $\overrightarrow{xy} \in J_{\text{ex}}^{K'}[V] \Leftrightarrow \overrightarrow{xy}^- \in J_{\text{ex}}^K[V]$, thus removing the 'twist': if for some edge xy of G_{ex} we add $\overrightarrow{xy}^-$ to J_{ex}^K then we add $\overrightarrow{xy}$ to $J_{\text{ex}}^{K'}$.

Lemma 2.13.

i. $\mathbb{P}^{t_{\rm hi}}(xy \in \Gamma) = d^*(\Gamma)/p$ *independently for each* $xy \in G^*$, *where* $\Gamma \in \{G_{\rm ex}, G^{gg'}_{ii'}, G'_i\}$, *and* $d^*(G_{\rm ex}) = 2p_{\rm ex}$, $d^*(G^{gg'}_i) = p^{gg'}_{ii'}$ *and* $d^*(G'_i) = p_{\max}$,

ii. each $\mathbb{P}^{t_{\rm hi}}(\overrightarrow{xw} \in \Psi) = d^*(\Psi)$ *for* $\Psi \in \{J^{\rm hi}, J^{\rm lo}_i, J^{\rm no}_i, J_{\rm ex}, J'_i\}$, *where* $d^*(J^g_i) = \alpha^g_i$ *for* $g \in \{\text{lo}, \text{no}\}$, $d^*(J^{\rm hi}) = \alpha_{\rm hi}$, $d^*(J_{\rm ex}) = p'_{\rm ex}$, $d^*(J'_i) = p_{\max}$,

iii. any subset $\mathcal{E}$ *of the events in (i) and (ii) is conditionally independent given any history of the algorithm at time* t_0 *if it has no pairs that are equivalent or mutually exclusive,*

iv. whp each Γ *as in (i) is* $(1.2\xi, s)$*-typical of density* $d(\Gamma) = d^*(\Gamma) \pm \Delta^{-.9}$,

v. for any $w \in W$, $u \in V(F)$, *distinct* $v_1, \dots, v_{s'} \in N_F(u)$ *with* $s' \le s$ *and* $x_1, \dots, x_{s'} \in V(G)$ *whp* $|N^-_{J_u}(w) \cap \bigcap_{i=1}^{s'} N^+_{\overrightarrow{G}_{uv_i}}(x_i)| = |A_u| \prod_{i=1}^{s'} (1 \pm 1.2\xi) p_{uv_i}$,

vi. in Case P, for all disjoint $S_-, S_+ \subseteq V$ *and* $R \subseteq W$ *each of size* $\le s$, *for any* $k, k_-, k_+ \in \{0, K'\}$, *writing* $\ell_0 = 7/8$ *and* $\ell_{K'} = 1/8$, *we have*

$$\left| N^-_{J^k_{\rm ex}}(R) \cap N^+_{J^{k_+}_{\rm ex}}(S_+) \cap N^-_{J^{k_-}_{\rm ex}}(S_-) \right| = (\ell_{k_-} p_{\rm ex})^{|S_-|} (\ell_{k_+} p_{\rm ex})^{|S_+|} (\ell_k p_{\rm ex})^{|R|} n \pm \eta^{.9} n.$$

Also $\left| W \cap N^+_{J_{\rm int}}(S_-) \cap N^+_{J^0_{\rm ex}}(S_+) \right|$ *is* $(p_{\rm ex}/8)^{|S_-|} (7p_{\rm ex}/8)^{|S_+|} n \pm \eta^{.9} n$ *if* $S_- \cup S_+$ *is* $3d$*-separated, or is* $\ge 2^{-3s} (p_{\rm ex}/8)^{|S_-|} (7p_{\rm ex}/8)^{|S_+|} n$ *if* (S_-, S_+) *is* $3d$*-separated,*

vii. whp each $|M'_{ai}|/M^a_i \in (1 - n^{-c'}, 1]$,

viii. whp all $d^\pm_{H^a_i}(x)$, $d^+_{J^a_i}(x)$ *and* $d^-_{J^a_i}(w)$ *are* $(1 \pm \delta)|A^a_i|$ *and* $N^-_{J^a_i}(w) \cap N^-_{J^{a'}_{i'}}(w) = H^a_i \cap H^{a'}_{i'} = \emptyset$ *whenever* $ai \ne a'i'$.

Proof. We start by briefly justifying statements (i–iv), which are fairly straightforward from the definition of the algorithm. The outcome of HIGH DEGREES determines G^* at time $t_{\rm hi}$, where $G \backslash G^*$ has maximum degree $< |A^*| \le 5n/\Delta$. For each xy independently, we include it in G_0 with probability p_0/p. Excluding $< 6dn$ such xy with $d(x,y) \le 3d$, the remainder have $\mathbb{P}(xy \in G_1) = 1 - p_0/p = p_1/p$. In DIGRAPH.iv each is then directed as $\overrightarrow{xy}$ or $\overrightarrow{yx}$ each with probability 1/2, and then in (vi) independently included in at most one Γ as in (i) with probability $d^*(\Gamma)/p_1$, so with overall probability $d^*(\Gamma)/p_1$. We note that xy may instead be included in H, again independently for all edges. This justifies statement (i), and then (iv) is immediate by typicality and Chernoff bounds.

For (ii), we start the calculation for each $\mathbb{P}^{t_{\rm hi}}(\overrightarrow{xw} \in \Psi)$ by multiplying $\overline{p}_w \pm \Delta^{-.9}$ for the event $\{x \in \overline{X}_w\}$ and then $p_1 = 1 - p_0$ for $\{\overrightarrow{xw} \notin J_0\}$. This gives $\overline{p}_w p_1$, which equals p_{xw} if w is not some $w(\overrightarrow{yx})$, and then we put $\overrightarrow{xw} \in \Psi$ with probability $d^*(\Psi)/p_{xw}$, giving an overall probability $d^*(\Psi)$. On the other hand, if w is some $w(\overrightarrow{yx})$ then we include $\overrightarrow{xw}$ in $J^{\rm hi}$ with probability $2\alpha_{\rm hi}/p_1\overline{p}_w$, so with overall probability $2\alpha_{\rm hi}$, or otherwise $\overrightarrow{xw}$ is available for other Ψ with probability $\overline{p}_w p_1 - 2\alpha_{\rm hi}$, which we define to be p_{xw} in this case, giving the same overall probabilities for $\mathbb{P}^{t_{\rm hi}}(\overrightarrow{xw} \in \Psi)$.

For (iii), we emphasise that we only have conditional independence given the history at time t_0, rather than independence, due to the dependence between $\{x \in \overline{X}_w\}$ and $\{y \in \overline{X}_w\}$ when $d(x,y) \le 3d$. This still suffices to prove concentration statements in two steps: first showing concentration of the conditional expectation under the random choices in INTERVALS, and then concentration under the random choices in DIGRAPHS. We illustrate this for (v), omitting the similar proof via Lemma 2.12 of (vi). For any $3d$-separated $Y \subseteq V$, for each $y \in Y$ independently we have $\mathbb{P}(y \in \overline{X}_w) = \overline{p}_w \pm \Delta^{-.9}$, so by Chernoff bounds whp $|Y \cap \overline{X}_w| = \overline{p}_w |Y| \pm 2\Delta^{-.9} n$. Then for each $y \in Y \cap \overline{X}_w$ we have $\mathbb{P}^{t_{\rm int}}(y \in N^-_{J_u}(w) \cap \bigcap_{i=1}^{s'} N^+_{\overrightarrow{G}_{uv_i}}(x_i)) = \alpha_u \overline{p}_w^{-1} \prod_{i=1}^{s'} p_{uv_i}$, By partitioning $\bigcap_{i=1}^{s'} N_G(x_i)$ into $3d$-separated sets we deduce (v) by a Chernoff bound.

For (vii), first note that for $\circ \in \{<\!\Lambda, \ge\!\Lambda\}$, if $m_\circ \ne 0$, then $m_\circ \ge \delta m$, so $p_\circ \ge \delta$. We also recall that $\sum_{(a,i) \in Q^\Lambda} M^a_i = m$. As B'_Λ is a union of m matchings each of size m, by [3, Theorem

2] of Barát, Gyárfás and Sárközy we have $|M'_\Lambda| \ge m - m^{.51}$, so $M^a_i - |M'_{ai}| \le m^{.51} < n^{-.4} M^a_i$ for each $(a,i) \in Q^\Lambda$. For $(a,i) \in Q^\Delta$ we apply Lemma 2.8 to $\mathcal{H}$ with $f(v^*w^*\ell_{a'i'j'}) = 1_{a'=a,i'=i}$. To see that this is valid, we take $D = m$, so each edge weight is D^{-1}, and note that each $\omega(\mathcal{H}[v^*])$ or $\omega(\mathcal{H}[w^*])$ is $m^{-1} \sum_{(a,i)\in Q^\Delta} M^a_i = 1$. Also, for any v^*w^* with some label ℓ_{aij} we have $\omega(\mathcal{H}[v^*w^*]) \le m^{-1} \lceil \Delta^{-.2}\Lambda/\delta \rceil < D^{1-c^2}$, say. Thus Lemma 2.8 gives $|M'_{ai}| = (1 \pm n^{-c'}) M^a_i$, recalling that $c' \ll c$, as required for (v).

For (viii), we analyse the construction of H, which is illustrated in Figure 4. The disjointness statements and $d^-_{J^a_i}(w) = d^+_{H^a_i}(\phi_w(a))$ are clear from the definition of the algorithm, so it remains to establish the degree estimates. It suffices to show all $d^\pm_{H^*_{ai}}(x) = (1 \pm .1\delta) p_1 \overline{p} m^a_i n/m$; indeed, for each $\overrightarrow{yx} \in H^*_{ai}$ we have $\overrightarrow{yx} \in H^a_i \Leftrightarrow \overrightarrow{yx} \in H$, where $\mathbb{P}(\overrightarrow{yx} \in H) = \alpha_{\text{hi}}/p_1(\overline{p} \pm d^{-.9})$ independently, so the estimates on $d^\pm_{H^a_i}(x)$ hold whp by Chernoff bounds.

Consider $d^-_{H^*_{ai}}(x)$ for $x \in U_h$, $h \in [m]$. It suffices to estimate the contribution from $V \setminus V_0$, as $|V_0| = n_0$ is negligible by comparison with the error term in the required estimate. Let $\circ \in \{<\Lambda, \ge\Lambda\}$ be such that $(a,i) \in Q^\circ$. For each $v^*w^* \in M'_{ai}$ we include in $M^h_\circ$ all edges of M^*_a with label ℓ_{aij} between V_{v^*+h} and W_{w^*+h}. There are $(1 \pm d^{-.8}) \overline{p} p_1 n/m$ such edges yw with $\overrightarrow{yx} \in \overrightarrow{G}_1$ and $x \in \overline{X}_w$ whp under the choices of $(V_{v*} : v^* \in V^*)$, intervals and orientation of G_1. For each such yw, in some component P of D^h_x, we have $\mathbb{P}(\text{hi}_P = \circ) = p_\circ$, independently for distinct P, so $\mathbb{E} d^-_{H^*_{ai}}(x) = p_\circ |M'_{ai}| \cdot (1 \pm 2d^{-.8}) \overline{p} p_1 n/m$. Under the orientation of G_1 whp each $|P| < \log^2 n$ by Chernoff bounds. Then $d^-_{H^*_{ai}}(x)$ is a $\log^2 n$-Lipschitz function of independent decisions of all hi_P, so Lemma 2.3 gives the required estimate on $d^-_{H^*_{ai}}(x)$.

Finally, consider $d^+_{H^*_{ai}}(y)$ for $y \in V(G)$. If $y \in V_0$, then there are exactly M^a_i values of $h \in [m]$ for which there is an edge $yw \in M^h_\circ$ with label ℓ_{aij}, some j. If $y \notin V_0$, then there are exactly M'_{ai} values of $h \in [m]$ for which there is an edge $yw \in M^h_\circ$ with label ℓ_{aij}, some j, as each $v^*w^* \in M'_{ai}$ satisfies $v^* = x_a + w^*$, determining some $h \in [m]$ such that $y \in V_{v^*+h}$, and some edge $yw \in M^h_\circ \cap M^*_a$, where $w \in W_{w^*+h}$. By typicality and Chernoff bounds whp each $|U_h \cap N^+_{\overrightarrow{G}_1}(y)| = (1 \pm 1.1\xi) p_1 n/m$. For each $x \in U_h \cap N^+_{\overrightarrow{G}_1}(y)$ independently $\mathbb{P}(\text{hi}_x = \circ) = p_\circ$, writing $\text{hi}_x = \text{hi}_P$ where P is the component of D^h_x containing y. The events $\{x \in \overline{X}_w\}$ are independent for distinct x in any $3d$-separated set, so by partitioning U_h into $3d$ such sets, applying a Chernoff bound to each, we obtain the required estimate on $d^+_{H^*_{ai}}(y)$, noting that it only depends on the number M^a_i or $|M'_{ai}|$ of $h \in [m]$ such that $M^h_\circ$ includes yw with some label ℓ_{aij}, and not the set of such h, which is yet to be determined when choosing the matchings M_a. □

3 Approximate decomposition

In this section we analyse the subroutine APPROXIMATE DECOMPOSITION, which applies hypergraph matchings to embed most of F in Cases S and P.

3.1 Hypergraph matchings

The main goal of this section is the following lemma, which will allow us to apply Lemma 2.8 to the hypergraph matchings chosen in APPROXIMATE DECOMPOSITION, i.e. all auxiliary vertices have ω'-weighted degree close to and not exceeding 1, and all ω'-weighted codegrees are small; statement (ii) concerns the degree that a pair ux would have if it were introduced as an auxiliary vertex (but we do not do this to avoid additional complications in analysing the relationship between ω' and ω). The 'bad' graphs and sets appearing in the lemma will be defined and analysed in Lemma 3.2.

Lemma 3.1. *whp for each* $i \in [i^*]$,

i. $\omega'(\mathcal{H}_i[\boldsymbol{v}]) \in (1-2\varepsilon^{.8}, 1]$ *for all* $\boldsymbol{v} \in V(\mathcal{H}_i)$,

ii. $\sum_{w \in W} \omega'(\text{“}\phi_w(u){=}x\text{”}) = 1 \pm 2\varepsilon^{.8}$ *for all* $x \in V(G)$, $u \in A_i$,

iii. $\omega'(\boldsymbol{e}) > (1+2\varepsilon_i)^{-1}\omega(\boldsymbol{e})$ *for all* $\boldsymbol{e} = \text{“}\phi_w(u){=}x\text{”}$ *with* $u \in A_i \setminus (A_w^{\rm bad} \cap N_{F'}(A^{\rm lo}))$, $\overrightarrow{xw} \in J_i \setminus J^{\rm bad}$,

iv. $\omega'(\mathcal{H}_i[uw]) \in (1-2\varepsilon_i, 1-.5\varepsilon_i]$ *for all* $u \in A_i \setminus (A_w^{\rm bad} \cap N_{F'}(A^{\rm lo}))$,

v. $\omega'(\mathcal{H}_i[\boldsymbol{v}\boldsymbol{v}']) < \Delta^{-.9}$ *for all* $\boldsymbol{v}, \boldsymbol{v}' \subseteq V(\mathcal{H}_i)$.

To satisfy the hypotheses of Lemma 2.8 for $(\mathcal{H}_i, \omega')$ we let $C = n$ and $\beta = c/2$. Then the edge weights satisfy $\omega'(\text{“}\phi_w(u){=}x\text{”}) \geq |A_u|^{-1} \geq C^{-1}$, the codegree condition holds by Lemma 3.1.v, and the vertex weights satisfy $\omega'(\mathcal{H}_i[\boldsymbol{v}]) = \sum\{\omega'(\boldsymbol{e}) : \boldsymbol{v} \in \boldsymbol{e}\} \leq 1 - .5\varepsilon_i$ by definition of ω'.

Next we define and bound the 'bad' graphs and sets appearing in Lemma 3.1. For each i write $A_i^{\rm lo} = A_i^{**} \cup A_i'$, where for $u \in A_i^{\rm lo}$ with $N_<(u) \cap A_0 = \{u'\}$ we include u in A_i^{**} if $u' \in A^* \cup A^{**}$ or in A_i' if $u' \in A_0'$. Let S_i^w be the multiset on A_0' where for each $u \in A_i'$, $N_<(u) \cap A_0 = \{u'\}$ we include $\phi_w(u')$, with multiplicity, so that $|S_i^w| = |A_i'|$. Note that all multiplicities in S_i^w are $\leq D$ by definition of A_0'. Let

$$J_i^{\rm bad} = \{\overrightarrow{xw} \in J_i^{\rm lo} : |S_i^w \cap N^-_{\overrightarrow{G}_1}(x)| \neq p_1|S_i^w| \pm \xi' n\} \quad \text{and} \quad J^{\rm bad} = \textstyle\bigcup_i J_i^{\rm bad},$$

$$B = \{x \in V(G) : d_{J^{\rm bad}}(x) > \delta^3 n\} \quad \text{and} \quad A_w^{\rm bad} := (A^{\rm lo} \cup A^{\rm no}) \cap \textstyle\bigcup_{x \in B} N_>(\phi_w^{-1}(x)).$$

Lemma 3.2. *whp* $|B| < \delta^4 n$ *and every* $d_{J^{\rm bad}}(w), |A_w^{\rm bad}| < \delta^3 n$.

Proof. We start by bounding $d_{J^{\rm bad}}(w)$ for each $w \in W$. We may assume $|A_0'| > \xi' n/D$, otherwise each $|S_i^w| \leq \xi' n$, and then $J^{\rm bad} = \emptyset$. As G is (ξ, s)-typical, a well-known non-partite variant of Lemma 2.5 implies that G is $\xi^{.1}$-regular. Writing $X = \phi_w(A_0')$, as $|X|, |\overline{X}_w \setminus X| \geq \xi' n/D$, standard regularity properties imply that $G[X, \overline{X}_w \setminus X]$ is $\xi^{.01}$-regular of density $p \pm \xi^{.01}$. Then Chernoff bounds imply that whp $\widetilde{G} = \{uv : u \in X, v \in \overline{X}_w \setminus X, \overrightarrow{uv} \in \overrightarrow{G}_1\}$ is $\xi^{.001}$-regular of density $p_1 \pm \xi^{.001}$. By Lemma 2.6 applied with $G = \widetilde{G}$ and $H = \{\{\phi_w(u')\} : u' \in A_0'\}$ we deduce $d_{J_i^{\rm bad}}(w) < \xi' n$, so $d_{J^{\rm bad}}(w) < \delta^7 n$, say. As $\delta^3 n|B| \leq \sum_{x \in B} d_{J^{\rm bad}}(x) \leq \sum_{w \in W} d_{J^{\rm bad}}(w) < |W|\delta^7 n$ we have $|B| < \delta^4 n$. Since $d_>(u) \leq p_{\max}^{-1}$ for every $u \in A_{\geq 1}$, we conclude that $|A_w^{\rm bad}| < p_{\max}^{-1}|\phi_w^{-1}(B)| < \delta^3 n$. □

Henceforth, we assume Lemmas 3.1 and 3.2 for all $i' < i$, our aim being to show that they hold for i. First we establish various properties of the matchings $\mathcal{M}_{i'}$ for $i' < i$ that will be used in the proof. We let $A_{i,w} = A_{t_i^+,w} \setminus A_{t_i,w}$, which is the set of $u \in A_i$ such that $\phi_w(u)$ is defined by the matching $\mathcal{M}_i$, and let $A_{i,w}^0 = A_i \setminus A_{i,w}$.

Lemma 3.3. *For all* $0 < i' \leq i$, $w \in W$, $W' \subseteq W$, $X \subseteq V(G)$, $U \subseteq A_{i'}^{\rm lo} \cup A_{i'}^{\rm no}$ *whp*

i. $|A_{i',w}^0| < 2.1\varepsilon_{i'}|A_{i'}|$,

ii. $|\{w \in W : \phi_w(u) \in X\}| < 1.1|X| + \Delta$ *for all* $u \in A_{i'}^{\rm lo} \cup A_{i'}^{\rm no}$,

iii. $|\{w \in W : \phi_w^{-1}(x) \in U\}| < 1.1|U| + \Delta$ *for all* $x \in V(G)$,

iv. $|\{w \in W : u \in A_w^{\rm bad}\}| < 5\delta^4 n$ *for all* $u \in A_{i'}^{\rm lo} \cup A_{i'}^{\rm no}$,

v. $.4\varepsilon_{i'}|W'| - \delta|A_u| < |\{w \in W' : u \in A_{i',w}^0\}| < 2.1\varepsilon_{i'}|W'| + \delta|A_u|$ *for all* $u \in A_{i'}$,

vi. $\sum_{u \in A_i^g} |F'[N_>(u), A_{i',w}^0]|$ *and* $|F'[A_i^g, A_{i',w}^0]|$ *are* $< 9p_{\max}^{-1}\varepsilon_{i'}|A_i^g|$ *for all* $g \in \{\text{hi}, \text{lo}, \text{no}\}$.

Proof. We write $|A_{i',w}| = f(\mathcal{M}_{i'})$, where f is the function on $\mathcal{H}_{i'}$ defined by $f(\text{“}\phi'_w(u')=x'\text{”}) = 1_{w'=w}$. We have $f(\mathcal{H}_{i'}, \omega') = \sum_{u' \in A_{i'}} \omega'(\mathcal{H}_{i'}[u'w]) \geq (1-2\varepsilon_{i'})|A_{i'}| - \delta^3 n$ by Lemmas 3.2 and 3.1.iv. For any $\boldsymbol{e} \in \mathcal{H}_{i'}$ we have $f_{\{\boldsymbol{e}\}}(\mathcal{H}_{i'}, \omega') \leq \omega'(\boldsymbol{e}) \leq 1 < C^{-\beta} f(\mathcal{H}_{i'}, \omega')$. By Lemma 2.8 whp $f(\mathcal{M}_{i'}) = (1 \pm C^{-\beta}) f(\mathcal{H}_{i'}, \omega') \geq 1 - 2.1\varepsilon_{i'}$, so $|A^0_{i',w'}| < 2.1\varepsilon_{i'}|A_{i'}|$.

Statements (ii) and (iii) are similar, using Lemma 3.1.ii. For (iv), we have $|\{w \in W : u \in A^{\text{bad}}_w\}| \leq \sum_{v \in N_<(u)} |\{w : \phi_w(v) \in B\}| \leq 4.4|B| + 4\Delta < 5\delta^4 n$ by (ii) and Lemma 3.2.

For (v) we write $|\{w \in W' : u \in A^0_{i',w}\}| = |W'| + \Delta^{.9} - f(\mathcal{M}_{i'})$ redefining f by setting $f(\emptyset) = \Delta^{.9}$ and each $f(\text{“}\phi_w(u')=x\text{”}) = 1_{w \in W', u'=u}$. Then $0 \leq \Delta^{.9} + \sum_{w \in W'} \omega'(\mathcal{H}_{i'}[uw]) - f(\mathcal{H}_{i'}, \omega') < |\{w \in W : u \in A^{\text{bad}}_w\}|$. By Lemma 3.1.iv we see that if $u \in A^{\text{hi}}_{i'}$, then $.5\varepsilon_{i'}|W'| < |W'| + \Delta^{.9} - f(\mathcal{H}_{i'}, \omega') < 2\varepsilon_{i'}|W'|$ and if $u \in A^{\text{lo}}_{i'} \cup A^{\text{no}}_{i'}$ then (iv) implies $.5\varepsilon_{i'}|W'| - 5\delta^4 n < |W'| + \Delta^{.9} - f(\mathcal{H}_{i'}, \omega') < 2\varepsilon_{i'}|W'| + 5\delta^4 n$, and all $f_{\{\boldsymbol{e}\}}(\mathcal{H}_{i'}, \omega') \leq 1$, so by Lemma 2.8 whp (ii) holds.

For (vi), we write $1 + |A^g_i|^{-1} \sum_{u \in A^g_i} |F'[A_{i',w}, N_>(u)]| = f(\mathcal{M}_{i'})$, where $f(\emptyset) = 1$ and each $f(\text{“}\phi_w(u')=x'\text{”}) = 1_{w=w'} |A^g_i|^{-1} \sum_{u \in A^g_i} |N_{F'}(u') \cap N_>(u)|$. Using Lemma 3.1.iv we have

$$f(\mathcal{H}_{i'}, \omega') - 1 > (1-2\varepsilon_{i'})|A^g_i|^{-1} \sum_{u \in A^g_i} \sum_{u' \in A_{i'}} |N_{F'}(u') \cap N_>(u)| = (1-2\varepsilon_{i'})|A^g_i|^{-1} \sum_{u \in A^g_i} |F'[A_{i'}, N_>(u)]|$$

and all $f_{\{\boldsymbol{e}\}}(\mathcal{H}_{i'}, \omega') \leq |A^g_i|^{-1} \max_{u'} \sum_{v \in N_>(u')} |N_<(v)| < 4p_{\max}^{-1} |A^g_i|^{-1} < C^{-\beta} f(\mathcal{H}_{i'}, \omega')$. By Lemma 2.8 whp $|A^g_i|^{-1} \sum_{u \in A^g_i} |F'[A^0_{i',w}, N_>(u)]| < 2.1\varepsilon_{i'}|A^g_i|^{-1} \sum_{u \in A^g_i} |F'[A_{i'}, N_>(u)]| \leq 9p_{\max}^{-1}\varepsilon_{i'}$. The second bound is similar so we omit the proof. □

We conclude this subsection by deducing Lemma 3.1 (for i) from the following estimates on ω-weighted degrees, which thus become the main goal of this section and which we will prove assuming Lemmas 3.2–3.4 for all $i' < i$.

Lemma 3.4. *whp for each $i \in [i^*]$,*

i. $\omega(\mathcal{H}_i[uw]) = 1 \pm \varepsilon_i$ *for all* $uw \in A_i \times W$,

ii. $\omega(\mathcal{H}_i[\overrightarrow{xw}])$ *is* $1 \pm .5\varepsilon^{.8}$ *for all* xw *in* J_i *and is* $1 \pm \varepsilon_i$ *if* $\overrightarrow{xw} \notin J^{\text{bad}}$,

iii. $\omega(\mathcal{H}_i[\overleftarrow{xy}])$ *is* $1 \pm \varepsilon^{.8}$ *for all* $\overleftarrow{xy} \in \overrightarrow{G}_{uv}$, *and is* $1 \pm \varepsilon_i$ *if* $v \notin A^{\text{lo}}$ *or* $y \notin B$,

iv. $\sum_{w \in W} \omega(\text{“}\phi_w(u)=x\text{”}) = 1 \pm \varepsilon_i$ *for all* $x \in V(G)$, $u \in A_i$.

Proof of Lemma 3.1 for i. We have already noted that all ω'-weighted degrees are $\leq 1 - .5\varepsilon_i$, so it remains to prove the lower bounds. Statements (i) and (ii) are immediate by applying the definition of ω', using Lemma 3.4, which gives $\omega'(\boldsymbol{e}) > (1+\varepsilon^{.8})^{-1}(1-.5\varepsilon_i)\omega(\boldsymbol{e})$ for any $\boldsymbol{e} \in \mathcal{H}_i$.

For (iii), if $\boldsymbol{e} = \text{“}\phi_w(u)=x\text{”}$ with $u \in A_i \setminus (A^{\text{bad}}_w \cap N_{F'}(A^{\text{lo}}))$, then $\omega(\mathcal{H}_i[\overleftarrow{xy}]) = 1 \pm \varepsilon_i$ for all $y = \phi_w(v)$, $v \in N_<(u)$ and if $\overrightarrow{xw} \in J_i \setminus J^{\text{bad}}$, then $\omega(\mathcal{H}_i[\overrightarrow{xw}]) = 1 \pm \varepsilon_i$ by Lemma 3.4, so $\omega'(\boldsymbol{e}) > (1+\varepsilon_i)^{-1}(1-.5\varepsilon_i)\omega(\boldsymbol{e}) > (1+1.6\varepsilon_i)^{-1}\omega(\boldsymbol{e})$ by definition.

For (iv), note that for any $\boldsymbol{e} \in \mathcal{H}_i$ containing uw with u in A^{hi}_i or $A^{\text{no}}_i \setminus A^{\text{bad}}_w$ we have $\omega'(\boldsymbol{e}) > (1+1.6\varepsilon_i)^{-1}\omega(\boldsymbol{e})$, so $\omega'(\mathcal{H}_i[uw]) > (1+1.6\varepsilon_i)^{-1}\omega(\mathcal{H}_i[uw]) > 1 - 2\varepsilon_i$. If instead $u \in A^{\text{lo}} \setminus A^{\text{bad}}_w$ then, by Lemma 3.1.iii, $\omega'(\mathcal{H}_i[uw])$ is at least the sum of $(1+1.6\varepsilon_i)^{-1}\omega(\boldsymbol{e})$ where $\boldsymbol{e} = \text{“}\phi_w(u)=x\text{”}$ over x with $\overrightarrow{xw} \notin J^{\text{bad}}$, so $\omega'(\mathcal{H}_i[uw]) > (1+1.6\varepsilon_i)^{-1}\omega(\mathcal{H}_i[uw]) - \delta^3 n \cdot (1 \pm .5\varepsilon^{.8})^{-1} p_{\max}^{-4}|A_u|^{-1} > 1 - 2\varepsilon_i$.

For (v), we consider codegrees according to the various types of vertices. First we note that each $\omega'(\text{“}\phi_w(u)=x\text{”}) \leq 2p_{\max}^{-4}|A_u|^{-1}$. Each $|A_u| \geq \Delta$, so this easily gives the codegree bound for the pairs appearing in the following bounds: $|\mathcal{H}_i[uw, vw]| = 0$, $|\mathcal{H}_i[\overrightarrow{xw}, \overrightarrow{yw}]| = 0$, $|\mathcal{H}_i[uw, \overrightarrow{xw}]| \leq 1$, $|\mathcal{H}_i[uw, \overleftarrow{xy}]| \leq 2$. If “$\phi_w(u)=x$” contains $\overrightarrow{xw}$ and $\overleftarrow{xy}$ then $u \in N_>(v)$ where $\phi_w(v) = y$, so $|\mathcal{H}_i[\overrightarrow{xw}, \overleftarrow{xy}]|$

is at most Δ, or at most $p_{\max}^{-1}$ if $v \in A_{\geq 1}$, or 0 if $v \in A_0$ and $\overrightarrow{xw} \in J^{\text{hi}}$. These weighted codegrees are therefore at most $2\Delta p_{\max}^{-4}(\delta n)^{-1}$ or $2p_{\max}^{-4}\Delta^{-1} < \Delta^{-.9}$.

It remains to bound $\omega'(\mathcal{H}_i[\overleftarrow{xy}, \overleftarrow{xy}'])$. Suppose $\overleftarrow{xy} \in \overrightarrow{G}_{u_0v_0}$ and $\overleftarrow{xy}' \in \overrightarrow{G}_{u_0v_0'}$, with $u_0v_0 \in F'[A_i, A_j]$ and $u_0v_0' \in F'[A_i, A_{j'}]$, say with $j' \leq j$. We have $j > 0$ as $|N_<(u) \cap A_0| \leq 1$ for all $u \in A_{\geq 1}$, and as $F[A^{\text{hi}}] = \emptyset$ we have $|A_{u_0}||A_{v_0}| \geq \delta n \Delta$.

Suppose first that $j' < j$. Let f be the function on $\mathcal{H}_j \cup \{\emptyset\}$, where $f(\emptyset) = \Delta^{-.99}$ and each $f(\text{“}\phi_w(v){=}z\text{”})$ is $2p_{\max}^{-4}|A_{u_0}|^{-1}$ if $z = y$, $v \in A_{v_0}$ and there are $u \in A_{u_0} \cap N_>(v)$ and $v' \in N_<(u) \cap A_{v_0'}$ with $\phi_w(v') = y'$, or 0 otherwise. If $j' > 0$, then for each $w \in W$ there is at most one v' with $\phi_w(v') = y'$, at most $p_{\max}^{-1}$ choices of $u \in A_i \cap N_>(v')$, and at most 4 choices of $v \in N_<(u)$, so $f(\mathcal{H}_j, \omega') \leq \Delta^{-.99} + 8np_{\max}^{-9}|A_{v_0}|^{-1}|A_{u_0}|^{-1} < 2\Delta^{-.99}$. If $j' = 0$, then there are at most Δ choices of $u \in A_i \cap N_>(v')$, and $u \in A^{\text{lo}}$, so every $v \in N_<(u)$ in A_{v_0} lies in A^{no}, so $f(\mathcal{H}_j, \omega') \leq \Delta^{-.99} + 8n\Delta p_{\max}^{-8}(\delta n)^{-2} < 2\Delta^{-.99}$. As each $f_{\{\boldsymbol{e}\}}(\mathcal{H}_j, \omega') \leq 2p_{\max}^{-4}\Delta^{-1} < C^{-\beta}f(\mathcal{H}_j, \omega')$, by Lemma 2.8 whp $\omega'(\mathcal{H}_i[\overleftarrow{xy}, \overleftarrow{xy}']) < f(\mathcal{M}_j) = (1 \pm C^{-\beta})f(\mathcal{H}_j, \omega') < \Delta^{-.9}$.

Now suppose $j' = j$. Then $j \neq 0$ and we cannot have v_0, v_0' both in A^{hi} by definition of A_0 as a span, so $|A_{v_0}||A_{v_0'}| \geq \delta n \Delta$. Let f be the function on $\binom{\mathcal{H}_j}{2} \cup \{\emptyset\}$, where $f(\emptyset) = \Delta^{-.99}$ and each $f(\text{“}\phi_w(v){=}z\text{”}, \text{“}\phi_w(v'){=}z'\text{”})$ is $2p_{\max}^{-4}|A_{u_0}|^{-1}$ if $z = y$, $z' = y'$, $v \in A_{v_0}$, $v \in A_{v_0'}$ and there is $u \in A_{u_0} \cap N_>(v) \cap N_>(v')$, or 0 otherwise. For each $u \in A_{u_0}$ there are ≤ 12 choices of (v, v') so $f(\mathcal{H}_j, \omega') \leq \Delta^{-.99} + 24np_{\max}^{-12}|A_{v_0}|^{-1}|A_{v_0'}|^{-1} < 2\Delta^{-.99}$. As above, by Lemma 2.8 whp $f(\mathcal{M}_j) < \Delta^{-.9}$. □

3.2 Potential embeddings

We define a hypergraph $\mathcal{H}$ with vertex parts G, $A_{\geq 1} \times W$ and $V(G) \times W$, which contains all potential edges of all $\mathcal{H}_i$, in the following sense. Given $w \in W$, $u \in V(T)$ and an injection $f : N_{\leq}(u) \to V(G)$ such that $f(u')f(u) \in G$ for all $u' \in N_<(u)$ we let $P_w(f)$ be the ‘potential edge’ containing $u \in V(T)$, $f(u) \in V(G)$ and $f(u')f(u) \in G$ for all $u' \in N_<(u)$. For any $u \in S \subseteq N_{\leq}(u)$ and injection $f' : S \to V(G)$ we let $P_w(f')$ be the set of all $P_w(f) \in \mathcal{H}$ such that f restricts to f' on S. We use the notation $P_w(u \to x)$ when $S = \{u\}$ with $f(u) = x$ and $P_w(uv \to \overleftarrow{xy})$ when $S = \{u, v\}$ with $f(u) = x$, $f(v) = y$.

For each time t we introduce a measure ω_t on $\mathcal{H}$ where each $\omega_t(P_w(f))$ estimates the probability given the history at time t that the w-embedding will be consistent with f. We define ω_t by the following formula involving other estimated probabilities that will be discussed below:

$$\omega_t(P_w(f)) = \omega_t^*(P_w : u \to x) \prod_{v \in N_<(u)} p_t(uv)^{-1}\omega_t^*(P_w : v \to f(v)).$$

The key parameter in this formula is $\omega_t^*(P_w : u \to x)$, which will estimate the probability $\mathbb{P}^t(\phi_w(u) = x)$ given the history at time t that we will have “$\phi_w(u){=}x$”. We also associate an edge probability $p_t(uv)$ to each $v \in N_<(u)$, where $p_t(uv) = 1$ if $t \geq t_v$, otherwise $p_t(uv)$ is p if $t < t_0$, is p_1 if $t_0 \leq t < t_1$, or is p_{uv} for $t \geq t_1$. The intuition for the formula is that conditional on “$\phi_w(u){=}x$”, the events $\phi_w(v) = f(v)$ become about $p_t(uv)^{-1}$ times more likely and are roughly independent. In our calculations it will be sufficient to work only with $\omega_t^*(P_w : u \to x)$, so the formula for the measure ω_t above can be thought of as just an intuitive explanation for why the calculations work (it is not logically necessary for the proof).

Note that we have introduced similar notation for two different quantities, namely $\omega_t^*(P_w : u \to x)$ and $\omega_t(P_w(u \to x)) = \sum\{\omega_t(P_w(f)) : P_w(f) \in P_w(u \to x)\}$; they will be approximately equal. In general, for any $u \in S \subseteq N_{\le}(u)$ and injection $f' : S \to V(G)$ we will have

$$\omega_t(P_w(f')) \approx \omega_t^*(P_w : f') := \omega_t^*(P_w : u \to x) \prod_{u' \in S \setminus \{u\}} p_t(uu')^{-1} \omega_t^*(P_w : u' \to f(u')).$$

Another important example of this will be

$$\omega_t(P_w(uv \to \overleftarrow{xy})) \approx \omega_t^*(P_w : uv \to \overleftarrow{xy}) = p_t(uv)^{-1} \omega_t^*(P_w : u \to x)\omega_t^*(P_w : u \to y).$$

Initially, we let all $\omega_0^*(P_w : u \to x) := n^{-1}$. (One can check that typicality of G gives $\omega_0(P_w(u \to x)) = (1 \pm \xi^{.9})\omega_0^*(P_w : u \to x)$.) For $t \ge t_{uw}$, i.e. times after $\phi_w(u)$ has been defined, we let $\omega_t^*(P_w : u \to x) := 1_{\text{“}\phi_w(u)=x\text{”}}$. We thus have $\omega_t(P_w(f')) = \omega_t^*(P_w : f')$ at times t after $\phi_w(u)$ has been defined for all $u \in N_{\le}(u) \setminus S$. In particular, if $t \ge t_{u'}$ for all $u' \in N_<(u)$ then there is at most one f with $f(u) = x$ consistent with the history, and we have $\omega_t(P_w(f)) = \omega_t^*(P_w : u \to x)$. Furthermore, for $u \in A_i$, when we come to step i of APPROXIMATE DECOMPOSITION we will have $\omega_{t_i}^*(P_w : u \to x) = \omega(\text{“}\phi_w(u)=x\text{”})$.

Now we define $\omega_t^*(P_w : u \to x)$ for general t and $u \in V(F)$. As mentioned above, we let $\omega_0^*(P_w : u \to x) = n^{-1}$ and $\omega_t^*(P_w : u \to x) = 1_{\text{“}\phi_w(u)=x\text{”}}$ for $t \ge t_{uw}$. At each time $t < t_{uw}$ where the possibility of “$\phi_w(u)=x$” depends on an event in the algorithm, if the event fails we let $\omega_t^*(P_w : u \to x) = 0$, and if it succeeds we will divide by an estimate for its probability, thus approximately preserving the conditional expectation of the surviving weight. We let $P_w^t(u \to \cdot)$ be the set of x such that $\omega_t^*(P_w : u \to x) \ne 0$ and define $P_w^t(\cdot \to x)$, $P_\cdot^t(u \to x)$ in analogy, and also define $P_w^t(\cdot \to \overleftarrow{xy})$ to be the set of $uv \in F'$ such that $\omega_t^*(P_w : u \to x) \ne 0$ and $\omega_t^*(P_w : v \to y) \ne 0$.

During HIGH DEGREES, when we embed any $u' \in N_<(u) \cap A^*$ we will have $\mathbb{P}^{t_{u'}^-}(\phi_w(u')x \in G) \approx p$, so if this occurs we let $\omega_{t_{u'}}^*(P_w : u \to x) := p^{-1}\omega_{t_{u'}^-}^*(P_w : u \to x)$. So at the end of HIGH DEGREES, $\omega_{t_{\text{hi}}}^*(P_w : u \to x)$ is $p^{-|N_<(u) \cap A^*|}n^{-1}$ for $x \in P_w^{t_{\text{hi}}}(u \to \cdot)$ or 0 otherwise. (Note that our estimate ignores the possibility that “$\phi_w(u)=x$” may be impossible due to requiring an edge of $G \setminus G^*$.)

In INTERVALS, we require $x \in \overline{X}_w$, which by Lemma 2.12.i occurs with probability $\approx \overline{p}_w$, and then we let $\omega_{t_{\text{int}}}^*(P_w : u \to x)$ be $\overline{p}_w^{-1}\omega_{t_{\text{hi}}}^*(P_w : u \to x)$ for $x \in P_w^{t_{\text{int}}}(u \to \cdot)$ or 0 otherwise.

In EMBED A_0, after choosing G_0 and J_0, there are two cases. If $u \in A_{\ge 1}$ we let $\omega_{t_{G_0}}^*(P_w : u \to x) = p_1^{-|N_<(u) \cap A_0 \setminus A^*|}\omega_{t_{\text{hi}}}^*(P_w : u \to x)$ for $x \in P_w^{t_{G_0}}(u \to \cdot)$ and 0 otherwise. Indeed (recalling $u \in A_{\ge 1}$), for any $x \in P_w^{t_{\text{int}}}(u \to \cdot)$ we have $x \in P_w^{t_{G_0}}(u \to \cdot)$ iff for $u' \in N_<(u) \cap A_0 \setminus A^*$ we have $x\phi_w(u') \in G_1$, which occurs with probability p_1. If $u \in A_0 \setminus A^*$ we require $\overrightarrow{xw} \in J_0$, which for available x has probability $p_0/\overline{p}_w$, and then we let $\omega_{t_{G_0}}^*(P_w : u \to x)$ be $p_0^{-1}\omega_{t_{\text{hi}}}^*(P_w : u \to x)$ for $x \in P_w^{t_{G_0}}(u \to \cdot)$ or 0 otherwise.

During the embedding of A_0, i.e. EMBED A_0 (ii), after choosing some $\phi_w(a)$ at time $t = t_a$, for $u \in A_0 \setminus A^*$ we let $\omega_t^*(P_w : u \to x)$ be $\omega_{t^-}^*(P_w : u \to x)$ if $au \notin T$, or if $au \in T$ we let $\omega_t^*(P_w : u \to x)$ be $p_0^{-1}\omega_{t^-}^*(P_w : u \to x)$ for $x \in P_w^t(u \to \cdot)$ or 0 otherwise. So $\omega_{t_0}^*(P_w : u \to x)$ is $p_0^{-|N_T(v) \cap A_0 \setminus A^*|}\omega_{t_{\text{hi}}}^*(P_w : u \to x)$ for $x \in P_w^{t_0}(u \to \cdot)$ or 0 otherwise. For $u \in A_{\ge 1}$, if $au \in F'$ we let $\omega_t^*(P_w : u \to x)$ be $p_1^{-1}\omega_{t^-}^*(P_w : u \to x)$ for $x \in P_w^t(u \to \cdot)$ or 0 otherwise.

For DIGRAPH, we let $\omega_{t_1}^*(P_w : u \to x)$ be $|A_u|^{-1}\prod_{v \in N_<(u) \cap A_0} p_{uv}^{-1}$ for $x \in P_w^{t_1}(u \to \cdot)$ or 0 otherwise. To justify this, we first consider $u \in A_i^a$, $a \in A_i^\Delta$, when $N_<(u) \cap A_0 = \emptyset$, and all $d_{H_i^a}^\pm(y) = (1 \pm \delta)|A_u|$ by Lemma 2.13.viii. If $u \in A_i^{\text{no}}$ we must choose $\overrightarrow{xw} \in J_i^{\text{no}}$ with probability $p_{xw}/\overline{p}_w \cdot \alpha_i^{\text{no}}/p_{xw} = \alpha_i^{\text{no}}/\overline{p}_w$. If instead $u \in A_i^{\text{lo}}$, we must choose $\overleftarrow{x\phi_w(v)}$ as an arc of $\overrightarrow{G}_{uv}$ for all edges

$x\phi_w(v)$ of G_1 with $v \in N_<(u) \cap A_0$, each with probability $\frac{1}{2} \cdot 2p_{uv}/p_1 = p_{uv}/p_1$ independently, and $\overrightarrow{wx} \in J_u$ with probability $\alpha_i^{\text{lo}}/\overline{p}_w$.

During APPROXIMATE DECOMPOSITION, we let $\omega_t^*(P_w : u \to x)$ be $|A_u|^{-1} \prod_{v \in N_<(u) \cap A_{t,w}} p_{uv}^{-1}$ for $x \in P_w^t(u \to \cdot)$, or 0 otherwise; we will see that whenever we embed $v \in N_<(u)$, we have $\mathbb{P}(\overleftarrow{x\phi_w(v)} \in \overrightarrow{G}_u) \approx p_{uv}$. We note that $\omega_{t_i}^*(P_w : u \to x) = \omega(\text{“}\phi_w(u){=}x\text{”})$, as mentioned above. We emphasise that the $\omega_t^*(P_w : u \to x)$ are definitions (with justifications provided only for intuition), and it is the sets $P_w^t(u \to x)$ which change during the algorithm.

We use the notation $\omega_t^*(P_w : u \to x)$ with a set of vertices in place of one or more of w, u, x to denote a sum of $\omega_t^*(P_w : u \to x)$ over these sets, for example $\omega_t^*(P_W : u \to x)$ or $\omega_t^*(P_w : V(T) \to x)$ or $\omega_t^*(P_W : F'[A_i, A_j] \to \overleftarrow{xy})$. To see the connection to weighted degrees in Lemma 3.4, observe that "$\phi_w(u){=}x$" $\in \mathcal{H}_i$ iff $x \in P_w^{t_i}(u \to \cdot)$, so

$$\omega(\mathcal{H}_i[uw]) = \sum \{\omega(\text{“}\phi_w(u){=}x\text{”}) : x \in P_w^{t_i}(u \to \cdot)\} = \omega_{t_i}^*(P_w : u \to N_{J_u}^-(w)) = \omega_{t_i}^*(P_w : u \to V(G))$$

and similarly $\omega(\mathcal{H}_i[\overrightarrow{xw}]) = \omega_{t_i}^*(P_w : A_{\overrightarrow{xw}} \to x)$ and $\omega(\mathcal{H}_i[\overleftarrow{xy}]) = \omega_{t_i}^*(P_W : F'[A_i^g, A_j^{g'}] \to \overleftarrow{xy})$ when $\overleftarrow{xy} \in G_{ij}^{gg'}$.

We conclude this subsection with the following lemma, which implies the estimate on the weighted degrees $\omega(\mathcal{H}_i[uw])$ in Lemma 3.4.i. The proof is immediate from Lemma 2.13.v, as $x \in P_w^t(u \to \cdot)$ for $t \geq t_1$ iff $x \in N_{J_u}(w) \cap \bigcap_{v \in N_<(u) \cap A_{t,w}} N^+_{\overrightarrow{G}_{uv}}(\phi_w(v))$.

Lemma 3.5. *For all $u \in A_{\geq 1}$, $w \in W$ and $t \geq t_1$ we have*

$$|P_w^t(u \to \cdot)| = (1 \pm 5\xi)\omega_t^*(P_w : u \to x)^{-1} = (1 \pm 5\xi)|A_u| \prod_{v \in N_<(u) \cap A_{t,w}} p_{uv}.$$

3.3 J degrees

In this subsection we prove the estimates in Lemma 3.4.v concerning weighted degrees $\omega(\mathcal{H}_i[\overrightarrow{xw}]) = \omega_{t_i}^*(P_w : A_{\overrightarrow{xw}} \to x)$ for $\overrightarrow{xw}$ in J_i. We start with the following estimate at time t_1.

Lemma 3.6. *For all $\overrightarrow{xw} \in J_i^{\text{hi}} \cup J_i^{\text{no}}$ we have $\omega_{t_1}^*(P_w : A_{\overrightarrow{xw}} \to x) = 1$. For all $\overrightarrow{xw} \in J_i^{\text{lo}}$ whp $\omega_{t_1}^*(P_w : A_{\overrightarrow{xw}} \to x) = 1 \pm .3\varepsilon^{.8}$.*

Proof. If $\overrightarrow{xw} \in J_i^{\text{hi}} \cup J_i^{\text{no}}$ then $P_w^{t_1}(\cdot \to x) = A_i^g$, where $g = \text{no}$ if $\overrightarrow{xw} \in J_i^{\text{no}}$ or $g = a$ if $\overrightarrow{xw} \in J_i^a$ for some $a \in A_i^\Delta$. We have $\omega_{t_1}^*(P_w : u \to x) = |A_i^g|^{-1}$ for all $u \in A_i^g$, so $\omega_{t_1}^*(P_w : A_i^g \to x) = 1$.

Now we consider the evolution of $\omega_t^*(P_w : A_i^{\text{lo}} \to x)$. Initially $\omega_0^*(P_w : A_i^{\text{lo}} \to x) = |A_i^{\text{lo}}|n^{-1} = \alpha_i^{\text{lo}}$. During HIGH DEGREES, for each $u \in A_i^{\text{lo}}$, when we embed some $a \in A^*$, if $a \notin N_<(u)$ we have $\omega_{t_a}^*(P_w : u \to x) = \omega_{t_a^-}^*(P_w : u \to x)$, whereas if $a \in N_<(u)$, as $\mathbb{P}^{t_a^-}(\phi_w(a) \in N_G(x)) = (1 \pm \delta)p$ by Lemma 2.10 we have

$$\mathbb{E}^{t_a^-} \omega_{t_a}^*(P_w : u \to x) = \mathbb{P}^{t_a^-}(\phi_w(a) \in N_G(x)) p^{-1} \omega_{t_a^-}^*(P_w : u \to x) = (1 \pm \delta)\omega_{t_a^-}^*(P_w : u \to x).$$

As each $|N_<(u)| \leq 4$ we have $\mathbb{E}^0 \omega_{t_{\text{hi}}}^*(P_w : A_i^{\text{lo}} \to x) = (1 \pm \delta)^4 \alpha_i^{\text{lo}}$. For concentration, we bound each $|\omega_{t_a}^*(P_w : A_i^{\text{lo}} \to x) - \omega_{t_a^-}^*(P_w : A_i^{\text{lo}} \to x)|$ by $\omega_{t_a^-}^*(P_w : A_i^{\text{lo}} \cap N_>(a) \to x) < \Delta p_{\max}^{-4}(\delta n)^{-1}$, so by Lemma 2.4 whp $\omega_{t_{\text{hi}}}^*(P_w : A_i^{\text{lo}} \to x) = (1 \pm 5\delta)\alpha_i^{\text{lo}}$.

After INTERVALS, we can assume $x \in \overline{X}_w$, and then $\omega_{t_{\text{int}}}^*(P_w : A_i^{\text{lo}} \to x) = \overline{p}_w^{-1} \omega_{t_{\text{hi}}}^*(P_w : A_i^{\text{lo}} \to x)$. After EMBED A_0, similarly to the above analysis for HIGH DEGREES, using Lemma

2.11 in place of Lemma 2.10, whp $\omega^*_{t_{**}}(P_w : A_i^{\rm lo} \to x) = (1 \pm 5D^{-.9})\omega^*_{t_{\rm int}}(P_w : A_i^{\rm lo} \to x)$ and $\omega^*_{t_0}(P_w : A_i^{\rm lo} \to x) = (1 \pm 5D^{-.9} \pm 5\alpha_0^{.9})\omega^*_{t_{\rm int}}(P_w : A_i^{\rm lo} \to x) = (1 \pm .1\varepsilon^{.8})\omega^*_{t_{\rm int}}(P_w : A_i^{\rm lo} \to x)$.

After DIGRAPH, the part J_i^g of J containing $\overrightarrow{xw}$ is determined; we can assume $g = \rm lo$, as we have already considered the other cases. Each $\omega^*_{t_1}(P_w : u \to x)$ is 0 unless $u \in A_i^{\rm lo}$ and we have the event E_u that $\overleftarrow{x\phi_w(u')}$ in $\overrightarrow{G}_{uu'}$ for the unique $u' \in N_<(u) \cap A_0$, in which case $\omega^*_{t_1}(P_w : u \to x) = |A_i^{\rm lo}|^{-1}p_{uu'}^{-1}$. By Lemma 2.13 we have

$$\mathbb{E}^{t_0}\omega^*_{t_1}(P_w : A_i^{\rm lo} \to x) = \sum_{u \in A_i^{\rm lo}} \mathbb{P}^{t_0}(E_u)|A_i^{\rm lo}|^{-1}p_{uu'}^{-1}$$

$$= \sum_{u \in A_i^{\rm lo}} (p_{uu'}/p_1) \cdot \omega^*_{t_0}(P_w : u \to x)\overline{p}_w(\alpha_i^{\rm lo})^{-1}(p_1/p_{uu'}) \pm \Delta^{-.9} = 1 \pm .2\varepsilon^{.8}.$$

For concentration, note that for each $x\phi_w(u') \in G^*$, the assignment in DIGRAPH affects $\omega^*_{t_1}(P_w : A_i^{\rm lo} \to x)$ by $\le |N_>(u') \cap A_i^{\rm lo}|p_{\max}^{-1}|A_i^{\rm lo}|^{-1} \le \Delta p_{\max}^{-1}(\delta n)^{-1}$. Thus by Lemma 2.4 whp $\omega^*_{t_1}(P_w : A_i^{\rm lo} \to x) = 1 \pm .3\varepsilon^{.8}$. □

Next we give a significantly better estimate for $\omega^*_{t_1}(P_w : A_i^g \to x)$ for $\overrightarrow{xw} \notin J_i^{\rm bad}$.

Lemma 3.7. *If $\overrightarrow{xw} \in J_i^{\rm lo} \setminus J_i^{\rm bad}$ then whp $\omega^*_{t_1}(P_w : A_i^{\rm lo} \to x) = 1 \pm \varepsilon_1$.*

Proof. By the proof of Lemma 3.6, whp $\omega^*_{t_0}(P_w : A_i^{**} \to x) = \omega^*_{t_{**}}(P_w : A_i^{**} \to x) = (1 \pm 6D^{-.9})(\overline{p}_w n)^{-1}|A_i^{**}| \pm \delta$, and it suffices to show $\omega^*_{t_0}(P_w : A_i' \to x) = (\overline{p}_w n)^{-1}|A_i'| \pm \delta$. For any $u \in A_i'$ we have $\omega^*_{t_0}(P_w : u \to x) = p_{uu'}/p_1 \cdot \omega^*_{t_{\rm int}}(P_w : u \to x) \cdot 1_{u' \in S_i^w \cap N^-_{\overrightarrow{G}_1}(x)}$, so by definition of $J_i^{\rm bad}$ we have $\omega^*_{t_0}(P_w : A_i' \to x) = (p_1\overline{p}_w n)^{-1}|S_i^w \cap N^-_{\overrightarrow{G}_1}(x)| = (\overline{p}_w n)^{-1}|A_i'| \pm \delta$. The lemma follows. □

Next we give an estimate that will be used in several further lemmas below. For any $U \subseteq V(T)\setminus A_0$ we let $\Gamma^2(U) = \{v : {\rm dist}_{T\setminus A_0}(v, U) \le 2\}$, where dist denotes graph distance. For $w \in W$ we let

$$U_w = \{u \in P_w^\tau(\cdot \to x) \cap \Gamma^2(A^{\rm lo}) : N_<(u) \cap A_w^{\rm bad} \ne \emptyset\}.$$

Lemma 3.8. *If $u \notin U_w$ and $u' \in N_<(u) \cap A_{i'}$ then $\sum_{x' \in N^-_{\overrightarrow{G}_{uu'}}(x)} \omega'(\text{“}\phi_w(u'){=}x'\text{”}) = (1 \pm 2.2\varepsilon_{i'})p_{uu'}$.*

Proof. We note that $\omega'(\text{“}\phi_w(u'){=}x'\text{”}) = (1 \pm 2\varepsilon_{i'})\omega(\text{“}\phi_w(u'){=}x'\text{”})$ for any $x' \in P_w^{t_{i'}}(u' \to \cdot)\setminus N_{J^{\rm bad}}(w)$. Indeed, this holds by Lemma 3.1.iii for i', as $u' \notin A_w^{\rm bad} \cap N_{F'}(A^{\rm lo})$ by definition of U_w. We deduce $\omega(\text{“}\phi_w(u'){=}x'\text{”}) = \omega^*_{t_{i'}}(P_w : u' \to x') = |A_{u'}|^{-1}\prod_{v \in N_<(u') \cap A_{<i'}} p_{u'v}^{-1} = ((1 \pm 5\xi)|P_w^{t_{i'}}(u' \to \cdot)|)^{-1}$ by Lemma 3.5, so

$$\sum_{x' \in N^-_{\overrightarrow{G}_{uu'}}(x)} \omega'(\text{“}\phi_w(u'){=}x'\text{”}) = (1 \pm 2.1\varepsilon_{i'})|P_w^{t_{i'}}(u' \to \cdot)|^{-1} \cdot |P_w^{t_{i'}}(u' \to \cdot) \cap N^-_{\overrightarrow{G}_{uu'}}(x)| \pm p_{\max}^{-4}\delta^2,$$

with $\pm p_{\max}^{-4}\delta^2$ accounting for $x' \in N_{J_{\rm bad}}(w)$ when $u' \notin A^{\rm hi}$, and so $\omega'(\text{“}\phi_w(u'){=}x'\text{”}) \le p_{\max}^{-4}(\delta n)^{-1}$. The lemma now follows by Lemma 2.13, similarly to the discussion before Lemma 3.5. □

Now we consider the evolution of $\omega^*_t(P_w : A_{\overrightarrow{xw}} \to x)$ during some step $i' < i$ of APPROXIMATE DECOMPOSITION. For lighter notation we write $\tau = t_{i'}$ and $\tau' = t_{i'}^+$.

Lemma 3.9. *whp $\omega^*_{\tau'}(P_w : A_{\overrightarrow{xw}} \to x) = (1 \pm \varepsilon_{i'}^{.9})\omega^*_\tau(P_w : A_{\overrightarrow{xw}} \to x)$.*

Proof. We consider the function ψ on $\binom{\mathcal{H}_{i'}}{\le 4}$, where $\psi(E_u)=0$ except if there is $u \in P^\tau_w(\cdot \to x) \cap A_{>i'} \setminus U_w$ such that E_u consists of disjoint edges "$\phi_w(u')$=x'" with $\overleftarrow{xx'} \in \overrightarrow{G}_{uu'}$ for each $u' \in N_<(u) \cap A_{i'}$, and then $\psi(E_u) = |A_u|^{-1} \prod_{u' \in N_<(u) \cap A_{\le i'}} p^{-1}_{uu'}$. Now, for $u \notin U_w$ with $N_<(u) \cap A^0_{i',w} = \emptyset$, given $x \in P^\tau_w(u \to \cdot)$, we have $x \in P^{\tau'}_w(u \to \cdot)$ iff $E_u \subseteq \mathcal{M}_{i'}$, and for such x we have $\psi(E_u) = \omega($"$\phi_w(u)$=x"$)$. Thus $\omega^*_{\tau'}(P_w : A_{\overrightarrow{xw}} \to x) = \sum\{\omega(\text{"}\phi_w(u)\text{=}x\text{"}) : x \in P^{\tau'}_w(u \to \cdot)\} = \psi(\mathcal{M}_{i'}) \pm \Delta_\psi \pm \Delta'_\psi$, where

$$\Delta_\psi = \sum\{\omega^*_{\tau'}(P_w : u \to x) : N_<(u) \cap A^0_{i',w} \neq \emptyset\} \le p^{-4}_{\max}|A_{\overrightarrow{xw}}|^{-1}|F'[A^0_{i',w}, A_{\overrightarrow{xw}}]| < 9\varepsilon_{i'} p^{-5}_{\max},$$
$$\Delta'_\psi = \sum_{u \in U_w} \omega^*_{\tau'}(P_w : u \to x) \le p^{-5}_{\max}|A^{\text{bad}}_w||A_{\overrightarrow{xw}}|^{-1} < p^{-5}_{\max}\delta^2.$$

Here we bounded Δ_ψ by Lemma 3.3.vi, and Δ'_ψ by Lemma 3.2, also assuming $|A_{\overrightarrow{xw}}| \ge \delta n$, as we may, as if $\overrightarrow{xw} \in J^{\text{hi}}$ then $U_w = \emptyset$, using $A^{\text{hi}} \cap \Gamma^2(A^{\text{lo}}) = \emptyset$ by the definition of A_0 as a 4-span.

Next we estimate

$$\psi(\mathcal{H}_{i'}, \omega') = \sum_{u \in P^\tau_w(\cdot \to x) \setminus U_w} |A_u|^{-1} \prod_{u'' \in N_<(u) \cap A_{\le i'}} p^{-1}_{uu''} \prod_{u' \in N_<(u) \cap A_{i'}} \sum_{x' \in N^-_{\overrightarrow{G}_{uu'}}(x)} \omega'(\text{"}\phi_w(u')\text{=}x'\text{"}).$$

By Lemma 3.8 we have

$$\begin{aligned}\psi(\mathcal{H}_{i'}, \omega') &= \sum_{u \in P^\tau_w(\cdot \to x) \setminus U_w} |A_u|^{-1} \prod_{u'' \in N_<(u) \cap A_{<i'}} p^{-1}_{uu''} \prod_{u' \in N_<(u) \cap A_{i'}} p^{-1}_{uu'}(1 \pm 2.2\varepsilon_{i'})p_{uu'} \\ &= (1 \pm 8.9\varepsilon_{i'})(\omega^*_\tau(P_w : A_{\overrightarrow{xw}} \to x) \pm p^{-5}_{\max}\delta^2),\end{aligned}$$

with the error term as in the estimate for Δ'_ψ. The lemma now follows from Lemma 2.8, noting that each $|E_u \cap E_{u'}| \le 1$ (otherwise u, u' would have two common neighbours), and for any $\boldsymbol{e}$ = "$\phi_w(u')$=x'" we have $f_{\{\boldsymbol{e}\}}(\mathcal{H}_{i'}, \omega') \le \sum_{u \in N_>(u')} 2|A_u|^{-1}p^{-4}_{\max} \le 2p^{-5}_{\max}\Delta^{-1} < C^{-\beta}\psi(\mathcal{H}_{i'}, \omega')$. □

Similarly to Lemma 2.11, we have the following estimates during the embedding of $A^0_{i',w}$ (which has size $< 2.1\varepsilon_{i'} n$ by Lemma 3.3).

Lemma 3.10. *For any $w \in W$, $W' \subseteq W$, $a \in A^0_{i',w} \cap A^{g'}_{i'}$, $x, y \in V(G)$, $i', i \in [i^*]$, $g, g' \in \{\text{hi}, \text{lo}, \text{no}\}$, writing A^w_a for the set of y such that $\phi_w(a) = y$ is possible given the history at time t^-_a, whp*

i. $\mathbb{P}^{t^-_a}(\phi_w(a) = y) = (1 \pm \varepsilon^{.9}_{i'})|A^w_a|^{-1}$,

ii. $\mathbb{P}^{t^-_a}(\phi_w(a) \in N^-_{\overrightarrow{G}^{gg'}_{ii'}}(x)) = (1 \pm \varepsilon^{.9}_{i'})p^{gg'}_{ii'}$,

iii. $|\{w \in W' : \phi_w(a) \in N^-_{\overrightarrow{G}^{gg'}_{ii'}}(x)\}| = (1 \pm \varepsilon^{.9}_{i'})p^{gg'}_{ii'}|W'| \pm n^{.8}$.

Proof. Recall that for each $a \in A_{i'}$ in any order, we let $W_a = \{w \in W : \phi_w(a) \text{ undefined}\}$, let $V_a \in \binom{V}{|W_a|}$ be uniformly random, and define $M_a = \{\phi_w(a)w : w \in W_a\}$=MATCH$(B_a, Z_a)$, where $Z_a = \{\phi_w(b)w : b \in N_<(a)\}$ and $B_a \subseteq V_a \times W_a$ consists of all vw with $v \in N_{J'_i}(w) \setminus \operatorname{Im}\phi_w$ and each $\overleftarrow{v\phi_w(b)}$ for $b \in N_<(a)$ an unused edge of $G'_{i'}$.

To justify the application of Lemma 2.7 in defining M_a, we first note that by Lemma 3.3.v, $|W_a| > .3\varepsilon_{i'} n$ for all $a \in A_{i'}$. Also Z_a has maximum degree ≤ 4. We also claim whp B_a is $\varepsilon^{.7}_{i'}$-super-regular of density $(1 \pm \varepsilon^{.7}_{i'})p^{|N_<(a)|+1}_{\max}$. To see this, we argue similarly to Lemmas 2.10 and 2.11, except that Lemma 2.6 is not applicable, so we instead apply Lemma 2.5. We have $|W_a| = |V_a| \ge .3\varepsilon_{i'} n$.

We let $G_{\rm free} \subseteq G'_{i'}$ denote the graph of unused edges, and let $\mathcal{B}$ be the bad event that $G'_{i'} \setminus G_{\rm free}$ has any vertex of degree $> .1\varepsilon_{i'}^{.9} n$. We will establish the claim at any step before $\mathcal{B}$ occurs, assuming the claim for any $b \prec a$, and deduce that whp $\mathcal{B}$ does not occur.

Consider any $R \in \binom{W_a}{\le 2}$. We have $N_{B_a}(R) = V_a \cap N^-_{J'_{i'}}(R) \cap N^+_{G_{\rm free}}\left(\bigcap_{w\in R} \phi_w(N_<(a))\right) \setminus \bigcup_{w \in R} \operatorname{Im}\phi_w$. As $\mathcal{B}$ does not occur, by Lemma 2.13 and a Chernoff bound whp $|N_{B_a}(R)| = (1 \pm \varepsilon_{i'}^{.8})(p_{\max}^{|N_<(a)|+1})^{|R|}|V_a|$, unless $R = \{w, w'\}$ with $\phi_w(N_<(a)) \cap \phi_{w'}(N_<(a)) \ne \emptyset$; by Lemma 2.8 there are whp $< n^{1.5}$ such pairs R.

Now consider any $R' \in \binom{V_a}{\le 2}$. Let W^t be the set of w such that $\phi_w(b) \in N^-_{G'_{i'}}(R')$ for all $b \in N_<(a)$ with $\phi_w(b)$ defined at time t. As $\mathcal{B}$ does not occur, $|N_{B_a}(R')| = |W_a \cap N^+_{J'_{i'}}(R') \cap W^{t_a^-}| \pm .5\varepsilon_{i'}^{.9} n$. For any $b \in N_<(a)$ and $w \in N^+_{J'_{i'}}(R')$, if $\phi_w(b)$ is defined by Lemma 2.7 during HIGH DEGREES or APPROXIMATE DECOMPOSITION then similarly to the proof of Lemma 2.11, writing $\tau = t_b^-$ and $\tau' = t_b$ whp $|W^{\tau'}| = |M^b[N^-_{G'_{i'}}(R'), W^\tau]| = (1 \pm \varepsilon_j^{.8}) p_{\max}^{|R'|}|W^\tau|$ by Lemma 2.7, assuming the claim for $b \in A_j$, $j < i'$.

Now suppose $\phi_w(b)$ is defined by the hypergraph matching in $\mathcal{H}_j$. We consider the function f on $\mathcal{H}_j$ with $f(\text{“}\phi_w(u){=}x\text{”}) = 1_{w \in W^\tau} 1_{u=b} 1_{x \in N^-_{G'_{i'}}(R')}$, so $f(\mathcal{H}_j, \omega') = \sum_{x \in N^-_{G'_{i'}}(R')} \sum_{w \in W^\tau} \omega'(\text{“}\phi_w(b){=}x\text{”})$. Similarly to the proof of Lemma 3.8, if $b \notin A_w^{\rm bad}$ then $\sum_{x\in N^-_{G'_{i'}}(R')} \omega'(\text{“}\phi_w(b){=}x\text{”}) = (1 \pm 2.2\varepsilon_j) p_{\max}^{|R'|}$. By Lemma 3.3.iv whp $|\{w : b \in A_w^{\rm bad}\}| < 5\delta^4 n$, so $f(\mathcal{H}_j, \omega') = \sum_{x \in N^-_{G'_{i'}}(R')} \sum_{w\in W^\tau} \omega'(\text{“}\phi_w(b){=}x\text{”}) = (1 \pm 2.3\varepsilon_j)|N^-_{G'_{i'}}(R')||W^\tau|$, and by Lemma 2.8 whp $|W^{\tau'}| = (1 \pm 3\varepsilon_j) p_{\max}^{|R'|}|W^\tau|$.

Together with Lemma 2.13.ii this proves the claim, and so justifies the definition of M_a. Statements (i–iii) of the lemma now follow directly from Lemma 2.7, considering $M_a[W', N^-_{\overrightarrow{G}^{gg'}_{ii'}}(x)]$ for (iii). Also, from (i) whp every vertex degree in $G'_{i'} \setminus G_{\rm free}$ is $(.1\varepsilon_{i'}^{.9} n, 4)$-dominated, so whp $\mathcal{B}$ does not occur. $\square$

We deduce the following estimate similarly to the proof of Lemma 3.6, using Lemma 3.10 in place of Lemma 2.11.

Lemma 3.11. *whp all* $\omega^*_{t_{i'+1}}(P_w : A_{\overrightarrow{xw}} \to x) = (1 \pm \varepsilon_{i'}^{.8})\omega^*_{t^+_{i'}}(P_w : A_{\overrightarrow{xw}} \to x)$.

We conclude by deducing the estimates on $\omega(\mathcal{H}_i[\overrightarrow{xw}]) = \omega^*_{t_i}(P_w : A_{\overrightarrow{xw}} \to x)$ required for Lemma 3.4. By Lemma 3.6 whp $\omega^*_{t_1}(P_w : A_{\overrightarrow{xw}} \to x) = 1 \pm .3\varepsilon^{.8}$, and by Lemmas 3.9 and 3.11 whp each $\omega^*_{t_{i'+1}}(P_w : A_{\overrightarrow{xw}} \to x) = (1 \pm 2\varepsilon_{i'}^{.8})\omega^*_{t^+_{i'}}(P_w : A_{\overrightarrow{xw}} \to x)$, so $\omega^*_{t_i}(P_w : A_{\overrightarrow{xw}} \to x) = 1 \pm .4\varepsilon^{.8}$. Also, if $\overrightarrow{xw} \notin J_i^{\rm bad}$ by Lemma 3.7 whp $\omega^*_{t_1}(P_w : A_{\overrightarrow{xw}} \to x) = 1 \pm \varepsilon_1$, and repeating the previous calculations gives $\omega^*_{t_i}(P_w : A_{\overrightarrow{xw}} \to x) = 1 \pm 3\varepsilon_{i-1}^{.8} = 1 \pm \varepsilon_i$. This completes the proof of Lemma 3.4.ii.

3.4 G degrees

This subsection concerns $\omega(\mathcal{H}_i[\overleftarrow{xy}])$ for $\overleftarrow{xy} \in \overrightarrow{G}_i$. We start by establishing Lemma 3.4.iii, which is Lemma 3.12 below, as this is needed for the analysis (and also for Lemma 4.2 below).

Lemma 3.12. *whp* $\omega^*_{t_i}(P_W : u \to x) = 1 \pm \varepsilon_i$ *for each* $x \in V(G)$ *and* $u \in A_i$.

We start with the corresponding estimate at time t_1.

Lemma 3.13. *whp* $\omega^*_{t_1}(P_W : u \to x) = 1 \pm \varepsilon_1$ *for each* $x \in V(G)$ *and* $u \in A_i$.

Proof. We consider cases according to the location of u.

If $u \in A_i^a$ with $a \in A_i^\Delta$ then $\omega^*_{t_1}(P_w : u \to x) = |A_u|^{-1} 1_{\phi_w(a)x \in H_i^a}$, so $\omega^*_{t_1}(P_W : u \to x) = |A_u|^{-1} d^-_{H_i^a}(x) = 1 \pm \varepsilon_1$ by Lemma 2.13.viii.

If $u \in A_i^{\rm no}$ then $\omega^*_{t_1}(P_w : u \to x) = |A_u|^{-1} 1_{\overrightarrow{xw} \in J_u}$, so by Lemma 2.13 and a Chernoff bound whp $\omega^*_{t_1}(P_W : u \to x) = 1 \pm \Delta^{-.1}$.

If $u \in A_i^{\rm lo}$ then $\omega^*_{t_1}(P_w : u \to x) = p_u^{-1} |A_u|^{-1} 1_{\overrightarrow{xw} \in J_u} 1_{\phi_w(u') \in N^-_{\overrightarrow{G}_{u0}}(x)}$, where $N_<(u) \cap A_0 = \{u'\}$. For each $x' \in N_{G^*}(x)$ there is a unique $w' \in W$ with $\phi_{w'}(u') = x'$, so $\omega^*_{t_1}(P_W : u \to x) = \sum_{x' \in N_{G^*}(x)} p_{u0}^{-1} |A_u|^{-1} 1_{\overleftarrow{xx'} \in \overrightarrow{G}_{u0}} 1_{\overrightarrow{xw'} \in J_u}$. As $|N_{G^*}(x)| = (1 \pm 2\xi)pn$, by Lemma 2.13 and a Chernoff bound whp $\omega^*_{t_1}(P_W : u \to x) = 1 \pm 3\xi$. □

Next we consider the evolution of $\omega^*_t(P_W : u \to x)$ at step $i' < i$ in the approximate decomposition, again writing $\tau = t_{i'}$, $\tau' = t^+_{i'}$.

Lemma 3.14. *whp* $\omega^*_{\tau'}(P_W : u \to x) = (1 \pm \varepsilon_{i'}^{.8}) \omega^*_\tau(P_W : u \to x)$.

Proof. Let $W' = \{w \in P^\tau_\cdot(u \to x) : u \notin U_w\}$. Consider the function ψ on $\binom{\mathcal{H}_{i'}}{\le 4}$ where $\psi(I) = 0$ except if there is $w \in W'$ such that I consists of disjoint edges "$\phi_w(u')$=x'" with $\overleftarrow{xx'} \in \overrightarrow{G}_{uu'}$ for each $u' \in N_<(u) \cap A_{i'}$, and then $\psi(I) = |A_u|^{-1} \prod_{u' \in N_<(u) \cap A_{\le i'}} p_{uu'}^{-1}$. Note that $\omega^*_{\tau'}(P_W : u \to x) = \psi(\mathcal{M}_{i'}) \pm \Delta_\psi \pm \Delta'_\psi$, where

$$\Delta_\psi = p_{\max}^{-4} |A_u|^{-1} \sum_{u' \in N_<(u) \cap A_{i'}} |\{w \in N_{J_u}(x) : u' \in A^0_{i',w}\}| < 4 p_{\max}^{-4} |A_u|^{-1} (2.1 \varepsilon_{i'} d_{J_u}(w) + \delta |A_u|) < \varepsilon_{i'}^{.9},$$

$$\Delta'_\psi = p_{\max}^{-4} |A_u|^{-1} |\{w \in W : u \in U_w\}| \le p_{\max}^{-4} (\delta n)^{-1} \sum_{u' \in N_<(u)} |\{w : u' \in A_w^{\rm bad}\}| < \delta.$$

Here we used Lemma 3.3 and Lemma 2.13.ii,viii to estimate Δ_ψ, and for Δ'_ψ we used Lemma 3.2.v and Lemma 3.3.iv, also noting that if $\Delta'_\psi \ne 0$ then $u, u' \notin A^{\rm hi}$ by definition of A_0, so $|A_u| \ge \delta n$. Finally

$$\begin{aligned}\psi(\mathcal{H}_{i'}, \omega') &= \sum_{w \in W'} |A_u|^{-1} \prod_{u' \in N_<(u) \cap A_{\le i'}} p_{uu'}^{-1} \prod_{u' \in N_<(u) \cap A_{i'}} \sum_{x' \in N^-_{\overrightarrow{G}_{uu'}}(x)} \omega'(\text{“}\phi_w(u')\text{=}x'\text{”}) \\ &= (1 \pm 9\varepsilon_{i'}) \omega^*_\tau(P_W : u \to x)\end{aligned}$$

by Lemma 3.8, and the lemma now follows from Lemma 2.8. □

We deduce Lemma 3.12 (i.e. Lemma 3.4.iv) from the previous two lemmas and the following estimate which holds similarly to Lemma 3.11, using Lemma 3.10.iii.

Lemma 3.15. *whp all* $\omega^*_{t_{i'+1}}(P_W : u \to x) = (1 \pm \varepsilon_{i'}^{.8}) \omega^*_{t^+_{i'}}(P_W : u \to x)$.

Now we turn to the degrees of $\overleftarrow{xy} \in \overrightarrow{G}_i$. We consider the evolution of $\omega^*_t(P_W : F'[A_i^g, A_j^{g'}] \to \overleftarrow{xy})$, where $\overleftarrow{xy} \in \overrightarrow{G}_{ij}^{gg'}$, $0 \le j < i$, and for convenient notation we label arcs of $F'[A_i^g, A_j^{g'}]$ as uv with $u \in A_i^g$, $v \in A_j^{g'}$. Recall that $\overrightarrow{G}_{i0}^{gg'} = \overrightarrow{G}_{i0}^g$ for all g'. We start with an estimate at time t_1.

Lemma 3.16. *whp* $\omega^*_{t_1}(P_W : F'[A^g_i, A^{g'}_j] \to \overleftarrow{xy}) = 1 \pm \varepsilon_1$ *for all* $\overleftarrow{xy} \in \overrightarrow{G}^{gg'}_{ij}$.

Proof. We consider cases according to g, g', i, j where $0 \le j < i$.

We start with the case $\overleftarrow{xy} \in \overrightarrow{G}^{\text{lo}}_{i0}$. For each $uv \in F'[A^{\text{lo}}_i, A_0]$, $w \in W$ we have $\omega^*_{t_1}(P_w : uv \to \overleftarrow{xy}) = 0$, except for the unique $w^v \in W$ with $\phi_{w^v}(v) = y$, for which $\omega^*_{t_1}(P_{w^v} : uv \to \overleftarrow{xy}) = \omega^*_{t_1}(P_{w^v} : u \to x) = (p^{\text{lo}}_{i0}|A^{\text{lo}}_i|)^{-1} 1_{xw^v \in J^{\text{lo}}_i}$, so $\omega^*_{t_1}(P_W : uv \to \overleftarrow{xy}) = (p^{\text{lo}}_{i0}|A^{\text{lo}}_i|)^{-1} 1_{xw^v \in J^{\text{lo}}_i}$. The w^v are distinct, so the events $\{xw^v \in J^{\text{lo}}_i\}$ are independent. Each affects $\omega^*_{t_1}(P_W : uv \to \overleftarrow{xy})$ by $< p^{-1}_{uv} p^{-1}_{\max} (\delta n)^{-1}$, so by Lemma 2.4 whp $\omega^*_{t_1}(P_W : F'[A^{\text{lo}}_i, A_0] \to \overleftarrow{xy}) = (p_{uv} n)^{-1} |F'[A^{\text{lo}}_i, A_0]| \pm n^{-.4} = p^{-1}_{uv}(p_{uv} - p_{\min}) \pm n^{-.4} = 1 \pm \varepsilon_1$.

Next consider the case $\overleftarrow{xy} \in \overrightarrow{G}^{\text{hi,no}}_{ij}$, $j \in [i-1]$. For each $uv \in F'[A^{\text{hi}}_i, A^{\text{no}}_j]$, $w \in W$ we have $\omega^*_{t_1}(P_w : uv \to \overleftarrow{xy}) = 0$, except if $\overrightarrow{xw} \in J_u$ and $\overrightarrow{yw} \in J_v$, when $\omega^*_{t_1}(P_w : uv \to \overleftarrow{xy}) = (p_{uv}|A_u||A_v|)^{-1}$. We have $\overrightarrow{xw} \in J_u$ iff $\phi_w(a) \in N^-_{H^a_i}(x)$, where $u \in A^a_i$, and as $d(x,y) > 3d$ since any close edges were removed from G_1, the events $\overleftarrow{xy} \in \overrightarrow{G}_{uv}$, $\{\overrightarrow{xw} \in J_u\}$ and $\{\overrightarrow{yw} \in J_v\}$ are conditionally independent given HIGH DEGREES. By Lemma 2.13 and a Chernoff bound whp there are $(1 \pm \Delta^{-.6}) \alpha_v d^-_{H^a_i}(x)$ choices of w with $\omega^*_{t_1}(P_w : uv \to \overleftarrow{xy}) \ne 0$, so $\omega^*_{t_1}(P_W : uv \to \overleftarrow{xy}) = (1 \pm \Delta^{-.6}) d^-_{H^a_i}(x) p^{-1}_{uv} |A_u|^{-1} n^{-1} = (1 \pm .2\varepsilon_1)(p_{uv} n)^{-1}$ by Lemma 2.13, giving the required estimate.

The case $\overleftarrow{xy} \in \overrightarrow{G}^{\text{no,hi}}_{ij}$, $j \in [i-1]$ is similar to the previous one.

Now consider the case $\overleftarrow{xy} \in \overrightarrow{G}^{\text{no,no}}_{ij}$, $j \in [i-1]$. For each $uv \in F'[A^{\text{no}}_i, A^{\text{no}}_j]$, $w \in W$ we have $\omega^*_{t_1}(P_w : uv \to \overleftarrow{xy}) = 0$, except if $\overrightarrow{xw} \in J_u$ and $\overrightarrow{yw} \in J_v$, when $\omega^*_{t_1}(P_w : uv \to \overleftarrow{xy}) = (p_{uv}|A_u||A_v|)^{-1}$. By a Chernoff bound whp $\omega^*_{t_1}(P_W : uv \to \overleftarrow{xy}) = \sum_{w \in W} \omega^*_{t_1}(P_w : uv \to \overleftarrow{xy}) = (p_{uv} n)^{-1} \pm n^{-1.4}$, giving the required estimate.

Finally, consider the case $\overleftarrow{xy} \in \overrightarrow{G}^{\text{lo,no}}_{ij}$, $j \in [i-1]$. (The case $\overleftarrow{xy} \in \overrightarrow{G}^{\text{no,lo}}_{ij}$ is similar, and this is the last case by the definition of A_0 as a 4-span.) For each $uv \in F'[A^{\text{lo}}_i, A^{\text{no}}_j]$, $w \in W$ we have $\omega^*_{t_1}(P_w : uv \to \overleftarrow{xy}) = 0$, except in the event E_{uvw} that $\overrightarrow{xw} \in J_u$, $\overrightarrow{yw} \in J_v$ and $\overleftarrow{xx'} \in \overrightarrow{G}^{\text{lo}}_{i0}$, where $x' = \phi(u')$, $\{u'\} = N_<(u) \cap A_0$, when $\omega^*_{t_1}(P_w : uv \to \overleftarrow{xy}) = (p_{uv} p^{\text{lo}}_{i0} |A_u||A_v|)^{-1}$. For each $x' \in V(G)$ there is a unique $w' \in W$ with $\phi_{w'}(u') = x'$, so $\omega^*_{t_1}(P_W : uv \to \overleftarrow{xy}) = \sum_{w \in W} \omega^*_{t_1}(P_w : uv \to \overleftarrow{xy}) = \sum_{x' \in N^-_{\overrightarrow{G}_1}(x)} 1_{E_{uvw}} (p_{uv} p^{\text{lo}}_{i0} |A_u||A_v|)^{-1}$. We have $\mathbb{E}^{t_0} \omega^*_{t_1}(P_W : uv \to \overleftarrow{xy}) = (p^{\text{lo}}_{i0}/p_1) d^-_{\overrightarrow{G}_1}(x) (p_{uv} p^{\text{lo}}_{i0} n^2)^{-1}$, where whp $d^-_{\overrightarrow{G}_1}(x) = (1 \pm 2\xi) p_1 n$. The decisions on $\overrightarrow{xw}$ and $\overrightarrow{yw}$ affect $\omega^*_{t_1}(P_W : uv \to \overleftarrow{xy})$ by $< (p_{uv} p^{\text{lo}}_{i0})^{-1} (p_{\max} \delta n)^{-2}$. For each $\overleftarrow{xx'}$, note that there are n choices of w', which determines $u' = \phi^{-1}_{w'}(x')$, then $< \Delta$ choices for each of u and v, so the decision on $\{\overleftarrow{xx'} \in G^{\text{lo}}_{i0}\}$ affects $\omega^*_{t_1}(P_W : uv \to \overleftarrow{xy})$ by $< n\Delta (p_{uv} p^{\text{lo}}_{i0})^{-1} (p_{\max} \delta n)^{-2}$. The required estimate now follows from Lemma 2.1. □

Next we consider the evolution of $\omega^*_t(P_W : F'[A^g_i, A^{g'}_j] \to \overleftarrow{xy})$ at step $i' < i$ in the approximate decomposition, again writing $\tau = t_{i'}$, $\tau' = t^+_{i'}$.

Lemma 3.17. *At step* $i' < i$ *whp* $\omega^*_{\tau'}(P_W : F'[A^g_i, A^{g'}_j] \to \overleftarrow{xy})$ *is* $(1 \pm .7\varepsilon^{.8}) \omega^*_\tau(P_W : F'[A^g_i, A^{g'}_j] \to \overleftarrow{xy})$ *for each* $0 \le j < i$ *and* $\overleftarrow{xy} \in \overrightarrow{G}^{gg'}_{ij}$, *and is* $(1 \pm \varepsilon^8_{i'}) \omega^*_\tau(P_W : F'[A^g_i, A^{g'}_j] \to \overleftarrow{xy})$ *if* $j \ne i'$ *or* $y \notin B$ *or* $\overleftarrow{xy} \notin G^{\text{no,lo}}$.

Proof. We start with the case $j < i'$. We note for each $uv \in F'[A^g_i, A^{g'}_j]$, $w \in W$ that $\omega^*_\tau(P_w : uv \to \overleftarrow{xy})$ and $\omega^*_{\tau'}(P_w : uv \to \overleftarrow{xy})$ are 0 unless $\phi_w(v) = y$, in which case $\omega^*_\tau(P_w : uv \to \overleftarrow{xy}) = \omega^*_\tau(P_w : u \to$

x), and $\omega^*_{\tau'}(P_w : uv \to \overleftarrow{xy}) = \omega^*_{\tau'}(P_w : u \to x)$. By Lemma 3.14 we deduce

$$\omega^*_{\tau'}(P_W : F'[A^g_i, A^{g'}_j] \to xy) = \sum_{u \in A^g_i} |N_<(u) \cap A^{g'}_j|(1 \pm \varepsilon^{.8}_{i'})\omega^*_\tau(P_W : u \to x)$$
$$= (1 \pm \varepsilon^{.8}_{i'})\omega^*_\tau(P_W : F'[A^g_i, A^{g'}_j] \to xy).$$

Now we may assume $j \geq i'$. Suppose $j = i'$. We consider the function ψ on $\binom{\mathcal{H}_{i'}}{\leq 4}$, where $\psi(E) = 0$ except if there are $w \in W$ and $uv \in P^\tau_w(\cdot \to \overleftarrow{xy}) \cap F'[A^g_i, A^{g'}_{i'}]$ with $u \notin U_w$ such that E consists of disjoint edges "$\phi_w(u')=x'$" with $\overleftarrow{xx'} \in \overrightarrow{G}_{uu'}$ for each $u' \in N_<(u) \cap A_{i'}$ and $x' = y$ when $u' = v$, and then $\psi(E) = \omega^*_{\tau'}(P_w : u \to x) = |A_u|^{-1} \prod_{u' \in N_<(u) \cap A_{\leq i'}} p^{-1}_{uu'}$.

Note that $\omega^*_{\tau'}(P_w : F'[A^g_i, A^{g'}_{i'}] \to \overleftarrow{xy}) = \psi(\mathcal{M}_{i'}) \pm \Delta_\psi \pm \Delta'_\psi$, where

$$\Delta_\psi = \sum_{uvw} \{\omega^*_{\tau'}(P_w : uv \to \overleftarrow{xy}) : N_<(u) \cap A^0_{i',w} \neq \emptyset\}$$
$$\leq \sum_{w : \overrightarrow{xw} \in J^g_i} \sum_{v \in A_{\overrightarrow{yw}}} |F'[A^0_{i',w}, N_>(v) \cap A^g_i]| p^{-8}_{\max} |A^g_i|^{-1} |A_{\overrightarrow{yw}}|^{-1} < 9\varepsilon_{i'} p^{-9}_{\max} d^+_{J^g_i}(x) |A^g_i|^{-1} < \varepsilon^{.9}_{i'},$$
$$\Delta'_\psi = \sum_{vw} \{\omega^*_{\tau'}(P_w : uv \to \overleftarrow{xy}) : u \in U_w\} \leq p^{-8}_{\max} n(\delta n)^{-2} \sum_{u' \in N_<(u)} |\{w : u' \in A^{\text{bad}}_w\}| < \delta^{.9}.$$

Here the bound on Δ_ψ follows from Lemmas 3.3.vi and 2.13, and the bound on Δ'_ψ by Lemmas 3.3.iv and 3.2, also noting that if $\Delta'_\psi \neq 0$ then $u, v \notin A^{\text{hi}}$ by definition of A_0, so $|A^g_i|, |A^{g'}_j| \geq \delta n$.

Next we estimate

$$\psi(\mathcal{H}_{i'}, \omega') = \sum_{uvw \in S, u \notin U_w} \left[|A_u|^{-1} \prod_{u'' \in N_<(u) \cap A_{\leq i'}} p^{-1}_{uu''} \; \omega'(\text{“}\phi_w(v)=y\text{”}) \prod_{u' \in N_<(u) \cap A_{i'} \setminus \{v\}} g_{uu'}(xw) \right],$$

where $S := \{uvw : w \in W, uv \in P^\tau_w(\cdot \to \overleftarrow{xy}) \cap F'[A^g_i, A^{g'}_j]\}$ and by Lemma 3.8

$$g_{uu'}(xw) := \sum_{x' \in N^-_{\overrightarrow{G}_{uu'}}(x)} \omega'(\text{“}\phi_w(u')=x'\text{”}) = (1 \pm 2.2\varepsilon_{i'}) p_{uu'}.$$

To obtain the required estimates in the case $j = i'$, by Lemma 2.8 it suffices to show that $\omega'(\text{“}\phi_w(v)=y\text{”})$ is $(1 \pm .6\varepsilon^{.8})\omega(\text{“}\phi_w(v)=y\text{”})$ when $v \in A^{\text{lo}}$ (which is equivalent to $\overleftarrow{xy} \in G^{\text{no,lo}}$) and that if $v \notin A^{\text{lo}}$ or $y \notin B$ then the sum of

$$f(uvw) := (|A_u||A_v|)^{-1} \prod_{u' \in N_<(u) \cap A_{\leq i'}} p^{-1}_{uu'} \prod_{v' \in N_<(v) \cap A_{\leq i'}} p^{-1}_{vv'}$$

over $uvw \in S$ for which $\overrightarrow{yw} \in J^{\text{bad}}$ is at most $\sqrt{\delta}$, the sum of $f(uvw)$ over $uvw \in S$ with $v \in A^{\text{bad}}_w \cap N_<(A^{\text{lo}})$ is at most $\sqrt{\delta}$, and for every other $uvw \in S$ we have $\omega'(\text{“}\phi_w(v)=y\text{”}) = (1 \pm 2\varepsilon_{i'})\omega(\text{“}\phi_w(v)=y\text{”})$.

So first assume $v \in A^{\text{lo}}$. For each $y' = \phi_w(v')$, $v' \in N_<(v)$, as $v \in A^{\text{lo}}$ we have $v' \notin A^{\text{lo}}$, so $\omega(\mathcal{H}_{i'}[\overleftarrow{yy'}]) = 1 \pm \varepsilon_{i'}$ by Lemma 3.4.iii for i'. Parts (i), (iii) imply that $\omega(\mathcal{H}_{i'}[\boldsymbol{v}]) = 1 \pm .5\varepsilon^{.8}$ for $\boldsymbol{v} = vw, \overrightarrow{yw}$, so $\omega'(\text{“}\phi_w(v)=y\text{”}) = (1 - .5\varepsilon_i)(1 \pm .5\varepsilon^{.8})\omega(\text{“}\phi_w(v)=y\text{”})$, as required.

Next, suppose $y \notin B$ or $v \notin A^{\text{lo}}$. Consider those $uvw \in S$ with $\overrightarrow{yw} \in J^{\text{bad}}$. Then $y \notin B$ and $v \in A^{\text{lo}}$, since otherwise $\overrightarrow{yw} \notin J^{\text{bad}}$. So $u \notin A^{\text{hi}}$ by the definition of A_0, and therefore $|A_u|, |A_v| \geq \delta n$. Since $y \notin B$, there are $< \delta^3 n$ choices of w with $\overrightarrow{yw} \in J^{\text{bad}}$, so the sum of $f(uvw)$ with $\overrightarrow{yw} \in J^{\text{bad}}$ is $\leq \delta^3 n \sum_{uv} p_{\max}^{-8} |A_u|^{-1} |A_v|^{-1} < \sqrt{\delta}$.

All other terms uvw have $\overrightarrow{yw} \notin J^{\text{bad}}$. Consider those $uvw \in S$ with $v \in A_w^{\text{bad}} \cap N_<(A^{\text{lo}})$. Then $v, u \in A^{\text{no}}$ by the definition of A_0 as a 4-span, so $|A_u|, |A_v| \geq \delta n$. Each remaining w has $< \delta^3 n$ choices of $v \in A_w^{\text{bad}}$ by Lemma 3.2, so the total sum of these terms $f(uvw)$ is $\leq \sum_w \delta^3 n p_{\max}^{-9} |A_u|^{-1} |A_v|^{-1} < \sqrt{\delta}$.

Since all other terms uvw have $\overrightarrow{yw} \notin J^{\text{bad}}$, by Lemma 3.4.iii for i', we may assume that there are (not necessarily distinct) $v', v'' \in N_<(v)$ with $\phi_w(v') \in B$ and $v'' \in A^{\text{lo}}$, or else we have $\omega'(\text{“}\phi_w(v){=}y\text{”}) = (1 - .5\varepsilon_{i'})(1 \pm \varepsilon_{i'})\omega(\text{“}\phi_w(v){=}y\text{”})$. But then $v \in A_w^{\text{bad}} \cap N_<(A^{\text{lo}})$, proving the claim and completing the case $j = i'$.

Finally, we suppose $i' < j < i$. We consider the function ψ on $\binom{\mathcal{H}_{i'}}{\leq 8}$, where $\psi(E) = 0$ except if there are $w \in W$ and $uv \in P_w^\tau(\cdot \to xy) \cap F'[A_i^g, A_j^{g'}]$ with $u \notin U_w$ such that E consists of disjoint edges “$\phi_w(u'){=}x'$” with $\overleftarrow{xx}' \in \overrightarrow{G}_{uu'}$ for each $u' \in N_<(u) \cap A_{i'}$ and $\overleftarrow{yx}' \in \overrightarrow{G}_{vv'}$ for each $v' \in N_<(v) \cap A_{i'}$, and then $\psi(E) = f(uvw)$. Note that $\omega_{\tau'}^*(P_W : F'[A_i^g, A_j^{g'}] \to \overleftarrow{xy}) = \psi(\mathcal{M}_{i'}) \pm \Delta_\psi \pm \Delta'_\psi$, with Δ_ψ as in the case $j = i'$ and

$$\Delta_{\psi'} = \sum_{uvw} \{\omega_{\tau'}^*(P_w : uv \to \overleftarrow{xy}) : N_<(v) \cap A_{i',w}^0 \neq \emptyset\} \leq \sum_{w : \overrightarrow{xw} \in J_i^g} p_{\max}^{-8} |A_i^g|^{-1} |A_{\overrightarrow{yw}}|^{-1} p_{\max}^{-1} |F'[A_{i',w}^0, A_{\overrightarrow{yw}}]|$$

$$< 9\varepsilon_{i'} p_{\max}^{-10} \sum_{w : \overrightarrow{xw} \in J_i^g} |A_i^g|^{-1} < \varepsilon_{i'}^{.9}$$

by Lemmas 3.3.vi and 2.13. Now we estimate

$$\psi'(\mathcal{H}_{i'}, \omega') = \sum_{uvw \in S, u \notin U_w} \left[f(uvw) \prod_{u' \in N_<(u) \cap A_{i'}} g_{uu'}(xw) \prod_{v' \in N_<(v) \cap A_{i'}} g_{vv'}(yw) \right].$$

By Lemma 3.8 each $g_{uu'}(xw) = (1 \pm 2.2\varepsilon_{i'}) p_{uu'}$ and $g_{vv'}(yw) = (1 \pm 2.2\varepsilon_{i'}) p_{vv'}$, so $\psi'(\mathcal{H}_{i'}, \omega') = (1 \pm \varepsilon_{i'}^{.9}) \omega_\tau^*(P_W : F'[A_i^g, A_j^{g'}] \to \overleftarrow{xy})$. The lemma now follows from Lemma 2.8. □

Similarly to Lemma 3.11, we also have the following estimate.

Lemma 3.18. *whp all* $\omega_{t_{i'+1}}^*(P_W : F'[A_i^g, A_j^{g'}] \to \overleftarrow{xy}) = (1 \pm \varepsilon_{i'}^{.8}) \omega_{t_{i'}^+}^*(P_W : F'[A_i^g, A_j^{g'}] \to \overleftarrow{xy})$.

We conclude the proof of Lemma 3.4 (and so of Lemma 3.1) by deducing the estimates on $\omega(\mathcal{H}_i[\overleftarrow{xy}]) = \omega_{t_i}^*(P_W : F' \to \overleftarrow{xy})$ required for Lemma 3.4. For any $\overleftarrow{xy} \in \overrightarrow{G}_{ij}^{gg'}$, by Lemma 3.16 whp $\omega_{t_1}^*(P_W : F'[A_i^g, A_j^{g'}] \to \overleftarrow{xy}) = 1 \pm \varepsilon_1$. At step $i' < i$, by Lemmas 3.17 and 3.18 whp $\omega_{\tau'}^*(P_W : F'[A_i^g, A_j^{g'}] \to \overleftarrow{xy})$ is $(1 \pm .7\varepsilon^{.8}) \omega_\tau^*(P_W : F'[A_i^g, A_j^{g'}] \to \overleftarrow{xy})$, and is $(1 \pm 3\varepsilon_{i'}^{.8}) \omega_\tau^*(P_W : F'[A_i^g, A_j^{g'}] \to \overleftarrow{xy})$ if $j \neq i'$ or $y \notin B$ or $\overleftarrow{xy} \notin G^{\text{no,lo}}$. Thus $\omega(\mathcal{H}_i[\overleftarrow{xy}]) = \omega_{t_i}^*(P_W : F'[A_i^g, A_j^{g'}] \to \overleftarrow{xy})$ is $1 \pm \varepsilon^{.8}$, and is $1 \pm 4\varepsilon_{i-1}^{.8} = 1 \pm \varepsilon_i$ if $j \neq i'$ or $y \notin B$ or $\overleftarrow{xy} \notin G^{\text{no,lo}}$.

4 Exact decomposition

In this section we complete the proof of our main theorem, in each of the cases S, P and L. We start in the first subsection with some properties of the leftover graph from the approximate decomposition required for cases S and P, then analyse each case separately over the following subsections.

4.1 Leftover graph

In both cases S and P the approximate decomposition constructs edge-disjoint copies F_w, $w \in W$ of $F = T \setminus P_{\rm ex}$. The leftover graph $G'_{\rm ex} = G \setminus \bigcup_{w\in W} \phi_w(F)$ is obtained from $G_{\rm ex}$ by adding all unused edges of $G \setminus G_{\rm ex}$ (and removing any orientations). We require the following typicality properties.

Lemma 4.1. *For any $w \in W$ and $S \in \binom{V(G)}{\le s}$ whp $|N^-_{J_{\rm ex}}(w) \cap G'_{\rm ex}(S)| = (1 \pm p_0^{.9})p'_{\rm ex}(2p_{\rm ex})^{|S|}n$.*

As in Lemma 2.13, a stronger form of this estimate holds with $G_{\rm ex}$ in place of $G'_{\rm ex}$, so it suffices to bound the maximum degree in the unused subgraph of $G \setminus G_{\rm ex}$. Given the trivial bounds whp $\Delta(G_0) < 1.1p_0n$ and $\Delta(G'_i) < 1.1p_{\max}n$, the following estimate implies Lemma 4.1.

Lemma 4.2. *whp the unused subgraph of each $G^{gg'}_{ii'}$ has maximum degree $< 5\varepsilon^{.8}n$.*

Proof. We fix $x \in V(G)$ and consider separately the contributions to the unused degree u_x of x from $N^{\pm}_{\vec{G}^{gg'}_{ii'}}(x)$. For indegrees, let f be the function on $\mathcal{H}_i$ defined by $f(\text{“}\phi_w(u)=x'\text{”}) = 1_{x'=x}1_{u\in A^g_i}|N_<(u)\cap A^{g'}_{i'}|$. We have $f(\mathcal{H}_i, \omega') = \sum\{\omega'(\mathcal{H}_i[\overleftarrow{xy}]) : y \in N^-_{\vec{G}^{gg'}_{ii'}}(x)\} \ge (1 - 2\varepsilon^{.8})d^{\pm}_{\vec{G}^{gg'}_{ii'}}(x)$ by Lemma 3.1, so by Lemma 2.8 whp this contribution to u_x is $< 2.1\varepsilon^{.8}n$.

For outdegrees, first note that if $i' = 0$ then for each $u \in A_0$ there is a unique $w \in W$ with “$\phi_w(u)=x$”, for which we use $|N_>(u) \cap A^g_i|$ out-arcs at x. Thus we use exactly $F'[A_0, A^g_i] = n(p^g_{i0} - p_{\max})$ out-arcs at x, so this contribution is whp $< 2p_{\max}n$. Now for $i' \in [i-1]$, let $f_{i'}$ be the function on $\mathcal{H}_{i'}$ defined by $f_{i'}(\text{“}\phi_w(u)=x'\text{”}) = 1_{x'=x}1_{u\in A^{g'}_{i'}}|N_>(u) \cap A^g_i|$. By Lemma 3.1 we have

$$f_{i'}(\mathcal{H}_{i'}, \omega') = \sum_{u\in A^{g'}_{i'}} |N_>(u) \cap A^g_i| \sum_{w\in W} \omega'(\text{“}\phi_w(u)=x\text{”}) \ge (1 - 2\varepsilon^{.8})|F'[A^{g'}_{i'}, A^g_i]|.$$

As $|F'[A^{g'}_{i'}, A^g_i]| = n(p^{gg'}_{ii'} - p_{\max})$, this contribution is whp $< 2.1\varepsilon^{.8}n$. □

4.2 Small stars

Here we conclude the proof of Theorem 1.1 in Case S, where $P_{\rm ex}$ is a union of leaf stars in $T \setminus T[A^*]$, each of size $\le \Lambda = n^{1-c}$, with $|P_{\rm ex}| = p_{\rm ex}n = p_-n/2 \pm n^{1-c}$. We start with some further properties of the approximate decomposition needed in this case.

Lemma 4.3.

i. For any $x \in V$ and $R \in \binom{V}{\le 2}$ whp $\Sigma := \sum_{w\in N_{J_{\rm ex}}(R)} d_{P_{\rm ex}}(\phi_w^{-1}(x)) = (1 \pm \varepsilon)(p'_{\rm ex})^{|R|}|P_{\rm ex}|$.

ii. For any $y \in V$ and $w \in W$ whp $\sum_{x\in G_{\rm ex}(y)} d_{P_{\rm ex}}(\phi_w^{-1}(x)) = (1 \pm \varepsilon)2p_{\rm ex}|P_{\rm ex}|$.

Proof. We prove (i) and omit the similar proof of (ii). We consider the contribution to Σ from each $a \in V(F)$ according to its location in T.

For each a in A_0 we define $M_a = \{\phi_w(a)w : w \in W\}$=MATCH$(B_a, Z_a)$. By Lemma 2.7, for each $b \in N_<(a)$ and $w \in W$ we have $\mathbb{P}^{t^-_b}(\phi_w(b) \in N_G(x)) = (1 \pm .1\xi')p$, and if $\phi_w(b) \in N_G(x)$ for all $b \in N_<(a)$ then $\mathbb{P}^{t^-_a}(\phi_w(a) = x) = (1 \pm .1\xi')(p^{|N_<(a)|}n)^{-1}$. By Lemma 2.4 whp the contribution to

Σ from A_0 is $\Sigma[A_0] := \sum_{a \in A_0} \sum_{w \in N_{J_{\rm ex}}(R)} 1_{\phi_w(a)=x} d_{P_{\rm ex}}(a) = (1 \pm .3\xi')|N_{J_{\rm ex}}(R)| \sum_{a \in A_0} d_{P_{\rm ex}}(a)/n = (1 \pm \xi')(p'_{\rm ex})^{|R|} \sum_{a \in A_0} d_{P_{\rm ex}}(a)$.

Now we consider the contribution from A_i with $i \in [i^*]$. By the proof of Lemma 3.12 whp $\sum_{w \in J_{\rm ex}(R)} \omega'(\text{“}\phi_w(u){=}x\text{”}) = \omega^*_{t_i}(P_{J_{\rm ex}(R)} : u \to x) = (1 \pm \varepsilon_i)(p'_{\rm ex})^{|R|}$ for each $i \in [i^*]$, $u \in A_i$. The function f on $\mathcal{H}_i$ defined by $f(\text{“}\phi_w(u){=}x'\text{”}) = 1_{x'=x} 1_{w \in N_{J_{\rm ex}}(R)} d_{P_{\rm ex}}(u)$ has

$$f(\mathcal{H}_i, \omega') = \sum_{u \in A_i} d_{P_{\rm ex}}(u) \sum_{w \in N_{J_{\rm ex}}(R)} \omega'(\text{“}\phi_w(u){=}x\text{”}) = (1 \pm \varepsilon_i)(p'_{\rm ex})^{|R|} \sum_{u \in A_i} d_{P_{\rm ex}}(u).$$

By Lemma 2.8 whp the contribution to Σ from the hypergraph matching embedding A_i is $\Sigma[A_i] := f(\mathcal{M}_i) = \sum_{u \in A_i} d_{P_{\rm ex}}(u)|\{w \in N_{J_{\rm ex}}(R) : \text{“}\phi_w(u){=}x\text{”} \in \mathcal{H}_i, u \notin A^0_{i,w}\}| = (1 \pm 2\varepsilon_i)(p'_{\rm ex})^{|R|} \sum_{u \in A_i} d_{P_{\rm ex}}(u)$.

It remains to consider the contribution from defining $\phi_w(a)$ for $w \in W_a$ by $\{\phi_w(a)w : w \in W_a\}$=MATCH$(B_a, Z_a)$, where $Z_a = \{\phi_w(b)w : b \in N_<(a)\}$ and $B_a \subseteq V_a \times W_a$ consists of all vw with $v \in N_{J'_i}(w) \setminus \operatorname{Im} \phi_w$ and each $\phi_w(b)v$ for $b \in N_<(a)$ an unused edge of G'_i. Here $V_a \in \binom{V}{|W_a|}$ is uniformly random, so $\mathbb{P}(x \in V_a) = |W_a|/n$. By Lemma 3.3.ii, $.3\varepsilon_i n < |W_a| < 2.2\varepsilon_i n$.

Similarly to the above analysis of A_0, whp there are $(1 \pm .1\xi')p_{\max}^{|N_<(a)|}|N_{J_{\rm ex}}(R)|$ choices of $w \in N_{J_{\rm ex}}(R)$ with $\phi_w(b) \in N_{G'_i}(x)$ for all $b \in N_<(a)$, and for each such w we have $\mathbb{P}^{t_a^-}(\phi_w(a) = x) = (1 \pm .1\xi')(p_{\max}^{|N_<(a)|}|W_a|)^{-1}$. Thus the contribution from defining $\phi_w(a)$ for $a \in A_i$, $w \in W_a$ is whp $\sum_{a \in A_i} d_{P_{\rm ex}}(a)|\{w \in W_a \cap N_{J_{\rm ex}}(R)\}|/n < 3.1\varepsilon_i \Sigma[A_i]$.

Summing all contributions gives the stated estimate. □

In the subroutine SMALL STARS we start by finding an orientation D of the leftover graph $G'_{\rm ex}$ such that each $d^+_D(x) = |L_x|$, where L_x is the set of all uw where u is a leaf of a star in $P_{\rm ex}$ with centre $\phi_w^{-1}(x)$. By the case $R = \emptyset$ of Lemma 4.3.i whp all $|L_x| = (1 \pm \varepsilon)|P_{\rm ex}|$. To construct D, we start with a uniformly random orientation of $G'_{\rm ex}$, and while not all $d^+_D(x) = |L_x|$, choose uniformly random x, y, z with $|L_x| > d^+_D(x)$, $|L_y| < d^+_D(y)$, $z \in N^+_D(y) \cap N^-_D(x)$ and reverse $\overrightarrow{yz}$, $\overrightarrow{zx}$.

To analyse this process, we first note that by typicality of $G'_{\rm ex}$ (Lemma 4.1) and a Chernoff bound, whp each $d^+_D(x) = (1 \pm 1.1p_0^{.7})p_{\rm ex}n = |L_x| \pm 2p_0^{.7}n$ and every $|N^+_D(y) \cap N^-_D(x)| \geq p_{\rm ex}^2 n/2$. Thus each vertex v plays the role of x or y at most $2p_0^{.7}n$ times. We let $\mathcal{B}$ be the bad event that we reverse $> .2p_0^{.6}n$ arcs at any vertex v. We will show that whp $\mathcal{B}$ does not occur. At any step before $\mathcal{B}$ occurs where we consider x and y as above, the number of choices for z is whp $> .49p_{\rm ex}^2 n - p_0^{.6}n > .48p_{\rm ex}^2 n$. Thus at any step v plays the role of z with probability $< 1/.48p_{\rm ex}^2 n$, so the number of such steps is $(\mu, 1)$-dominated with $\mu < 2p_0^{.7}n^2/.48p_{\rm ex}^2 n < .1p_0^{.6}n$. By Lemma 2.4 we deduce that whp $\mathcal{B}$ does not occur, so we can construct D with all $d^+_D(x) = |L_x|$.

Now for each $x \in V(G)$ in arbitrary order, we define $\phi_w(u)$ for all $uw \in L_x$ by $M_x = \{\{uw, \phi_w(u)\} : uw \in L_x\}$=MATCH$(F_x, \emptyset)$, where $F_x \subseteq L_x \times N^+_D(x)$ consists of all $\{uw, y\}$ with $uw \in L_x$, $y \in N^+_D(x) \cap N_{J_{\rm ex}}(w) \setminus \operatorname{Im} \phi_w$.

To analyse this process, we consider $Z_x = \{\{uw, y\} \in L_x \times (N^+_D(x) \cap J_{\rm ex}(w) \cap \operatorname{Im} \phi_w)\}$ and let $\mathcal{B}_x$ be the bad event that Z_x has any vertex of degree $> .1p_{\rm ex}^{.9}|L_x|$. Recall that $p_{\rm ex} \ll p'_{\rm ex} \ll 1$ in Case S and at the beginning of SMALL STARS we have $\operatorname{Im} \phi_w \cap N_{J_{\rm ex}}(w) = \emptyset$.

Lemma 4.4. *whp under the construction of D, if $\mathcal{B}_x$ does not occur then F_x is $p_{\rm ex}^{.02}$-super-regular of density $(1 \pm p_{\rm ex}^{.8})p'_{\rm ex}$.*

Proof. Any $R \in \binom{N^+_D(x)}{\leq 2}$ has $|N_{F_x}(R)| = \sum_{w \in N_{J_{\rm ex}}(R)} d_{P_{\rm ex}}(\phi_w^{-1}(x)) \pm .1|R|p_{\rm ex}^{.9}|L_x| = (1 \pm .1p_{\rm ex}^{.8})(p'_{\rm ex})^{|R'|}|P_{\rm ex}|$ by Lemma 4.3. Any $R' \in \binom{L_x}{\leq 2}$ has $|N_{F_x}(R')| = |N^+_D(x) \cap \bigcap_{uw \in R'} J_{\rm ex}(w)| \pm .1|R'|p_{\rm ex}^{.9}|L_x|$, which by

Lemma 4.1 and a Chernoff bound is whp $(1 \pm .1p_{\text{ex}}^{.8})(p'_{\text{ex}})^{|R'|}|P_{\text{ex}}|$ unless $R' = \{uw, u'w\}$ for some w; there are $< \sum_{w \in W} d_{P_{\text{ex}}}(\phi_w^{-1}(x))^2 < n^{2-c}$ such R'. The lemma now follows from Lemma 2.5. □

By Lemma 2.7 we can choose $M_x = \{\{uw, \phi_w(u)\} : uw \in L_x\}$=MATCH$(F_x, \emptyset)$, and $\mathbb{P}(\phi_w(u) = y) = (1 \pm p_{\text{ex}}^{.01})(p'_{\text{ex}}|P_{\text{ex}}|)^{-1}$ for all $\{uw, y\} \in F_x$.

It remains to show whp no $\mathcal{B}_x$ occurs. We define a stopping time τ as the first x for which $\mathcal{B}_x$ occurs and bound $\mathbb{P}(\tau = x)$.

First we bound $d_{Z_x}(uw)$ for $uw \in L_x$. For any $y \in N_D^+(x)$, when processing any x' before x we defined $\phi_w(u')$ for $d_{P_{\text{ex}}}(\phi_w^{-1}(x'))$ leaves u' of $\phi_w^{-1}(x')$, each of which could be y if $y \in N_D^+(x') \cap J_{\text{ex}}(w)$, with probability $< (.9p'_{\text{ex}}|P_{\text{ex}}|)^{-1}$. Thus $d_{Z_x}(uw)$ is (μ, n^{1-c})-dominated with $\mu = \sum_{x'} |N_D^+(xx') \cap J_{\text{ex}}(w)| d_{P_{\text{ex}}}(\phi_w^{-1}(x'))(.9p'_{\text{ex}}|P_{\text{ex}}|)^{-1} < 1.2p_{\text{ex}}^2 n$, so by Lemma 2.4 whp $d_{Z_x}(uw) < 3p_{\text{ex}}|L_x|$.

Now we bound $d_{Z_x}(y)$ for $y \in N_D^+(x)$. For any $uw \in L_x$ with $w \in J_{\text{ex}}(y)$, when processing any $x' \in N_D^-(y)$ before x, we had $\phi_w(u) = y$ for some leaf u with probability $< d_{P_{\text{ex}}}(\phi_w^{-1}(x'))(.9p'_{\text{ex}}|P_{\text{ex}}|)^{-1}$. Thus $d_{Z_x}(y)$ is (μ, n^{1-c})-dominated with $\mu = (.9p'_{\text{ex}}|P_{\text{ex}}|)^{-1} \sum_{uw \in L_x} 1_{w \in J_{\text{ex}}(y)} \sum_{x' \in N_D^-(y)} d_{P_{\text{ex}}}(\phi_w^{-1}(x')) < (.9p'_{\text{ex}}|P_{\text{ex}}|)^{-1} \cdot |L_{\text{ex}}| \cdot (1+\varepsilon)p_{\text{ex}}|P_{\text{ex}}|$ by Lemma 4.3.ii, so whp $d_{Z_x}(w) < 3|L_x|p_{\text{ex}}/p'_{\text{ex}}$.

Thus whp no $\mathcal{B}_x$ occurs, as required.

4.3 Paths

Here we conclude the proof of Theorem 1.1 in Case P, where P_{ex} is the vertex-disjoint union of two leaf edges in $T \setminus T[A^*]$ and $p_+ n/101K$ bare $8K$-paths in $T \setminus T[A^*]$.

The first phase of the PATHS subroutine fixes parity, as follows. We call $x \in V(G)$ *odd* if the parity of $d_{G'_{\text{ex}}}(x)$ differs from that of the number of w such that $x = \phi_w(a)$ where a is the end of a bare path in P_{ex}. We let X be the set of odd vertices and $a_1\ell_1$, $a_2\ell_2$ be the leaf edges in P_{ex}, with leaves ℓ_1, ℓ_2.

First we define all $\phi_w(\ell_1)$ by $M_1 = \{\phi_w(\ell_1)w : w \in W\}$=MATCH$(B_1, Z_1)$, where $Z_1 = \{\phi_w(a_1)w\}_{w \in W}$ and $B_1 = \{vw : v \in N_{J_{\text{ex}}}(w), v\phi_w(\ell_1) \in G_{\text{free}}\}$. Lemma 2.7 applies, as Z_1 is a matching and similarly to the proof of Lemma 3.10 whp B_1 is $p_0^{.5}$-super-regular with density $(1 \pm p_0^{.5})p'_{\text{ex}}p_{\text{ex}}$. Similarly, Lemma 2.7 applies to justify the definition of $\phi_w(\ell_2)$ for $w \in W'$ by $M'_2 = \{\phi_w(\ell_2)w : w \in W'\}$=MATCH$(B'_2, Z'_2)$ and $\phi_w(\ell_2)$ for $w \in W \setminus W'$ by $M_2 = \{\phi_w(\ell_2)w : w \in W \setminus W'\}$=MATCH$(B_2, Z_2)$. By construction, there are no odd vertices after the embeddings of ℓ_1 and ℓ_2.

Next for each $w \in W$ we need $8d(x,y)$-paths P_w^{xy} in P_{ex} for each $[x,y] \in \mathcal{Y}_w$ centred in vertex-disjoint bare $(8d(x,y)+2)$-paths in P_{ex}. We greedily choose these paths within the bare $8K$-paths in P_{ex} that exist by definition of Case P. By Lemma 2.12, the total number of vertices required by these paths is $\sum\{8d(x,y)+2 : [x,y] \in \mathcal{Y}_w\} = 8|Y_w| + |\mathcal{Y}_w| = (1-\eta)|P_{\text{ex}}| \pm nd^{-.9}$. At most $d^{-.9}|P_{\text{ex}}|$ vertices of the bare $8K$-paths cannot be used due to rounding errors, so as $d^{-1} \ll \eta$ the algorithm to choose all P_w^{xy} can be completed.

Now we extend each ϕ_w to an embedding of $P_{\text{ex}} \setminus \bigcup_{xy} P_w^{xy}$ so that $\phi_w^{-1}(x)$, $\phi_w^{-1}(y^+)$ are the ends of P_w^{xy}, according to a random greedy algorithm, where in each step, in any order, we define some $\phi_w(a) = z$, uniformly at random with $z \in J_{\text{ex}}(w) \setminus \operatorname{Im}\phi_w$ and $zz' \in G_{\text{free}}$ whenever $z' = \phi_w(b)$ with $b \in N_T(a)$. Writing E for the set of ends of paths in P_{ex}, for any vertex y we use $|\{w : \phi^{-1}(y) \in E\}| < 1.1|E| < |P_{\text{ex}}|/3K$ edges at y due to it playing the role of an end.

Let X_y be the number of additional edges used at y during the random greedy algorithm, and let $\mathcal{B}$ be the bad event that any $X_y > .1\eta^{.9}n$. We claim whp $\mathcal{B}$ does not occur. To see this, consider any step before $\mathcal{B}$ occurs, and suppose we are defining $\phi_w(a)$. Let R be the set of $b \in N_T(a)$ such that $\phi_w(b)$ has been defined and note that $|R| \le 2$. By Lemma 4.1 there are $((1 \pm p_0^{.9})p_{\text{ex}})^{|R|}p'_{\text{ex}}n$ choices of $z \in J_{\rm ex}(w) \cap N_{G'_{\rm ex}}(R)$, of which we forbid $< 2\eta p_{\rm ex} n$ in $\mathrm{Im}\,\phi_w$ and $< \eta^{.9} n$ if $\mathcal{B}$ has not occurred. As $\eta = \eta_+ \ll p_+$ (and $p_+ \le 13 p_{\rm ex}$) we can choose z, and any z is chosen with probability $< (.9 p_{\rm ex}^2 p'_{\rm ex} n)^{-1}$. Thus X_y is $(\mu, 2)$-dominated with $\mu = (.9 p_{\rm ex}^2 p'_{\rm ex} n)^{-1} \sum_{w \in J_{\rm ex}(y)} |P_{\rm ex} \setminus \bigcup_{xy} P^{xy}_w| < 1.2 \eta p_{\rm ex}^{-1} n$, so by Lemma 2.4 whp $< .1\eta^{.9} n$, which proves the claim.

Thus the random greedy algorithm can be completed, and the remaining graph $G_{\rm free}$ is an $\eta^{.9}$-perturbation of $G_{\rm ex}$, i.e. $|G_{\rm ex}(x) \triangle G_{\rm free}(x)| < \eta^{.9} n$ for any $x \in V$. By construction, every $d_{G_{\rm free}}(x)$ is even. The following lemma will complete the proof of Theorem 1.1 in Case P.

Lemma 4.5. *One can decompose $G_{\rm free}$ into $(G_w : w \in W)$ such that each G_w is a vertex-disjoint union of $8d(x, y)$-paths $\phi_w(P^{xy}_w)$ between x and y^+ for $[x, y] \in \mathcal{Y}_w$, internally disjoint from $\mathrm{Im}\,\phi_w$.*

The proof of Lemma 4.5 is similar to the corresponding arguments in [20], so we will be brief and give more details only where there are significant differences. We require the following result on wheel decompositions; see [20] for its derivation from [18] and discussion of how it provides the required paths. The statement requires a few definitions. An 8-wheel consists of a directed 8-cycle (called the rim), another vertex (called the hub), and an arc from each rim vertex to the hub. We obtain the special 8-wheel $\overrightarrow{W}_8^K$ by giving all arcs colour 0 except that one rim edge $\overrightarrow{xy}$ and one spoke $\overrightarrow{yw}$ have colour K.

Theorem 4.6. *Let $n^{-1} \ll \delta \ll \omega \ll 1$, $s = 2^{50 \cdot 8^3}$ and $d \ll n$. Let $J = J^0 \cup J^K$ be a digraph with arcs coloured 0 or K, with $V(J)$ partitioned as (V, W) where $\omega n \le |V|, |W| \le n$, such that all arcs in $J[V, W]$ point towards W and $J[W] = \emptyset$. Then J has a $\overrightarrow{W}_8^K$-decomposition such that every hub lies in W if the following hold:*

Divisibility: *$d^-_J(w) = 8 d^-_{J^K}(w)$ for all $w \in W$, and for all $v \in V$ we have $d^-_J(v, V) = d^+_J(v, V) = d^+_J(v, W)$ and $d^-_{J^K}(v, V) = d^+_{J^K}(v, W)$.*

Regularity: *each $3d$-separated copy of $\overrightarrow{W}_8^K$ in J has a weight in $[\omega n^{-7}, \omega^{-1} n^{-7}]$ such that for any arc $\overrightarrow{e}$ there is total weight $1 \pm \delta$ on wheels containing $\overrightarrow{e}$.*

Extendability: *for all disjoint $A, B \subseteq V$ and $L \subseteq W$ each of size $\le s$, for any $a, b, \ell \in \{0, K\}$ we have $|N^+_{J^a}(A) \cap N^-_{J^b}(B) \cap N^-_{J^\ell}(L)| \ge \omega n$, and furthermore, if (A, B) is $3d$-separated then $|N^+_{J^0}(A) \cap N^+_{J^K}(B) \cap W| \ge \omega n$.*

Proof of Lemma 4.5. Recall that we constructed $J_{\rm ex}$ in DIGRAPH, such that for every $xy \in G_{\rm ex}$, we have exactly one of $\overrightarrow{xy} \in J^0_{\rm ex}$, $\overrightarrow{yx} \in J^0_{\rm ex}$, $\overrightarrow{xy}^- \in J^K_{\rm ex}$, $\overrightarrow{yx}^- \in J^K_{\rm ex}$, and there are also $\overrightarrow{yw} \in J^0_{\rm ex}[V, W]$. Add the arcs $J^K_{\rm ex}[V, W] = \{\overrightarrow{xw} : x \in Y_w\}$. It suffices to find an $\eta^{.6}$-perturbation L of $J_{\rm ex}$, i.e. L is obtained from $J_{\rm ex}$ by adding, deleting or recolouring at most $\eta^{.6} n$ arcs at each vertex, where $L[V]$ corresponds to $G_{\rm free}$ under twisting, and each $N^-_L(w) \subseteq V \setminus \mathrm{Im}\,\phi_w$, and a set E of edge-disjoint copies of $\overrightarrow{W}_8^K$ in L, such that Theorem 4.6 applies to give a $\overrightarrow{W}_8^K$-decomposition of $L' := L \setminus \bigcup E$. This will suffice, by taking each G_w to consist of the union of the 8-paths that correspond under twisting to the rim 8-cycles of the copies of $\overrightarrow{W}_8^K$ containing w. Here an arc of L corresponds to an edge $xy \in G_{\rm free}$ under twisting if it is $\overrightarrow{xy} \in L^0$ or $\overrightarrow{yx} \in L^0$ or $\overrightarrow{xy}^- \in L^K$ or $\overrightarrow{yx}^- \in L^K$ (which is a more flexible notion than in [20], as it does not depend on the orientation of L.)

Whenever we make a series of γn^2 modifications to $J_{\rm ex}$ of some type which involves changing edges at some intermediate vertex z, we always ensure that no vertex plays the role of z more than $\gamma \eta^{-.1} n$ times. There will always be more than, say, $2\eta^{.1} n$ valid choices of z, by Lemmas 2.12 and 2.13, and thus we can avoid the set of at most $\eta^{.1} n$ overused vertices. This series of modifications will add $\gamma \eta^{-.1}$ to the perturbation constant.

We start by deleting arcs corresponding to $G_{\rm ex} \setminus G_{\rm free}$, adding arcs $\overrightarrow{xy}$ for each $xy \in G_{\rm free} \setminus G_{\rm ex}$, replacing any $\overrightarrow{xy}$ of colour K where $d(x,y) < 3d$ with $\overrightarrow{xy}^+$ of colour 0 and deleting arcs $\overrightarrow{yw}$ in $L[V,W]$ with $y \in N^-_{J_{\rm ex}}(w) \cap \operatorname{Im}\phi_w$. Next we delete or add arbitrary arcs $\overrightarrow{yw}$ with $y \in V \setminus (\operatorname{Im}\phi_w \cup (Y_w)^+ \cup N^-_{J_{\rm ex}}(w))$ until each $d^-_L(w) = 8|Y_w|$, and so $|L[V,W]| = |L[V]|$. We require $< \eta^{.9}n$ such arcs for each w, by Lemma 2.12 and the bounds on X_y during the embedding of $P_{\rm ex} \setminus \bigcup_{xy} P^{xy}_w$.

While $|L^0[V]| > |L^0[V,W]|$ we replace some $\overrightarrow{xy} \in L^0[V]$ by $\overrightarrow{xy}^- \in L^K[V]$, or while $|L^0[V]| < |L^0[V,W]|$ we replace some $\overrightarrow{xy} \in L^K[V]$ by $\overrightarrow{xy}^+ \in L^0[V]$, continuing until $|L^0[V,W]| = |L^0[V]|$, and hence $|L^K[V,W]| = |L^K[V]|$.

Next we balance degrees in L^K. While there are x,y in V with $d^-_{L^K}(x,V) > d^+_{L^K}(x,W)$ and $d^-_{L^K}(y,V) < d^+_{L^K}(y,W)$, we choose $z \in V$ such that $\overrightarrow{zx} \in L^K$, $\overrightarrow{zy}^+ \in L^0$ and replace these arcs by $\overrightarrow{zx}^+ \in L^0$, $\overrightarrow{zy} \in L^K$. While there are x,y in V with $d^+_{L^K}(x,V) > d^+_{L^K}(x,W)$ and $d^+_{L^K}(y,V) < d^+_{L^K}(y,W)$, we choose $z \in V$ such that $\overrightarrow{xz} \in L^K$, $\overrightarrow{yz}^+ \in L^0$ and replace these arcs by $\overrightarrow{yz} \in L^K$, $\overrightarrow{xz}^+ \in L^0$. We continue until every $d^+_{L^K}(v,V) = d^-_{L^K}(v,V) = d^+_{L^K}(v,W)$.

Now we require some new modifications which do not appear in [20]. We start by noting that each $d_L(x,V)$ is even. To see this, note that as $L[V]$ corresponds to $G_{\rm free}$ under twisting we have $d_L(x,V) = d_{G_{\rm free}}(x) + d^-_{L^K}(x^-,V) - d^-_{L^K}(x,V) = d_{G_{\rm free}}(x) + d^+_{L^K}(x^-,W) - d^+_{L^K}(x,W) = d_{G_{\rm free}}(x)$ where the last equality follows from interval properties (listed before the definition of INTERVALS). While there are $x,y \in V$ with $d_L(x,V) < 2d^+_L(x,W)$ and $d_L(y,V) > 2d^+_L(y,W)$, we add $\overrightarrow{yw}$ to L^0 and remove $\overrightarrow{xw}$ from L^0 for some $w \in N^+_{L^0}(x,W) \setminus N^+_L(y,W)$ with $y \notin \operatorname{Im}\phi_w$. We continue until every $d_L(v,V) = 2d^+_L(v,W)$.

While there are x,y in V with $d^+_{L^0}(x,V) > d^+_{L^0}(x,W)$ and $d^+_{L^0}(y,V) < d^+_{L^0}(y,W)$, we choose $z \in V$ such that $\overrightarrow{xz} \in L^0$, $\overrightarrow{yz} \in L^0$ and replace these arcs by $\overrightarrow{zx} \in L^0$, $\overrightarrow{zy} \in L^0$. Now every $d^+_L(v,V) = d^+_L(v,W)$. Thus L satisfies the required divisibility conditions, and is an $\eta^{.6}$-perturbation of $J_{\rm ex}$, and $L[V]$ corresponds to $G_{\rm free}$ under twisting. It remains to satisfy the extendability and regularity conditions of Theorem 4.6. A summary of the argument is as follows (we omit the details as they are very similar to those in [20]). There are many wheels on each arc, so we can greedily cover all $\overrightarrow{xy} \in L[V]$ with edge-disjoint wheels, incurring an insignificant perturbation of L. A stronger version of the extendability hypothesis with $J_{\rm ex}$ in place of L holds by Lemmas 2.12 and 2.13, and so it holds for the perturbation L. By typicality, the regularity condition is satisfied by assigning the same weight $\hat{W}$ to every wheel, choosing $\hat{W}$ so that any arc is in $\approx \hat{W}^{-1}$ wheels. □

4.4 Large stars

Here we conclude the proof of Theorem 1.1 in Case L, where all but at most p_+n vertices of T belong to leaf stars of size $\geq \Lambda = n^{1-c}$. The argument is self-contained: there is no approximate step, and the entire embedding is achieved by the subroutine LARGE STARS.

We start by letting $\mathcal{S}$ be the union of all maximal leaf stars in T that have size $\geq \Lambda$. We let $F = T \setminus \mathcal{S}$; by assumption $|V(F)| \leq p_+n$. We let $S^+ = \{v \in V(T) : d_T(v) \geq \Lambda\}$, so that $|S^+| < 2\Delta$ and $S \subseteq S^+ \subseteq V(F)$, where S is the set of star centres of $\mathcal{S}$.

We partition W as $W_1 \cup W_2 \cup W_3$ with each $||W_i| - n/3| < 1$. For each $v \in V(G)$, we independently choose at most one of $\mathbb{P}(v \in U^a_i) = d_{\mathcal{S}}(a)/3|\mathcal{S}|$ with $a \in S$, $i \in [3]$. By Chernoff bounds, whp each $|U^a_i| = nd_{\mathcal{S}}(a)/3|\mathcal{S}| \pm n^{.9}$. We let $U_i = \bigcup_a U^a_i$. While $\sum_{i=1}^3 ||W_i| - |U_i|| > 0$, we relocate a vertex so as to decrease this sum, thus relocating $< n^{.9}$ to or from any U^i_a, so $< 3\Delta n^{.9} < n^{.99}$ in total.

Noting that F is a tree, we can fix an order $\prec$ on $V(F)$ such that $N_<(u) = \{v \prec u : vu \in F\} = \{u^-\}$ has size 1 for all $u \neq u_0 \in V(F)$. We fix distinct $\phi_w(u_0)$, $w \in W$ with $\phi_w(u_0) \in U_i$ whenever

$w \in W_i$. We construct edge-disjoint copies F_w of F by considering $a \in F_w$ in $\prec$ order, defining all $\phi_w(a)$ by $M_i^a = \{\phi_w(a)w : w \in W_i\}$=MATCH$(B_i^a, Z_i^a)$, $i \in [3]$, and updating

$$G_{\text{free}} = \{\text{unused edges}\}, \quad Z = \{vw : v \in V(F_w)\}, \text{ and}$$
$$J = \{\overrightarrow{xx}' : x = \phi_w(a), x' \in Z(w) \cap U^a\}_{w \in W, a \in S}.$$

By construction $G \setminus G_{\text{free}}$ and Z both have maximum degree $\leq |V(F)| \leq p_+ n$.

Lemma 4.7. *Every edge is used at most once and J has no 2-cycles.*

Proof. First note that as each $B_i^a(w) \subseteq G_{\text{free}}(\phi_w(a^-)) \setminus Z(w)$ we embed each $\phi_w(a)$ to a vertex not yet used by F_w so that $\phi_w(a^-)\phi_w(a)$ is an unused edge. Furthermore, when $a \in S$, for each $\overleftarrow{xx}' \in J$, by excluding $\phi_w(a) \in N_J^+(Z(w) \cap U^a)$ we do not add $\overrightarrow{xx}'$ to J due to $x = \phi_w(a)$, $x' \in Z(w) \cap U^a$, and by excluding $N_J^-(\phi_w(b)) \cap U^b$ where $x' \in U^b$ we do not add $\overrightarrow{xx}'$ to J due to $x = \phi_w(b)$, $x' = \phi_w(a) \in Z(w) \cap U^b$. As before, by including all $\phi_w(a^-)w$ in Z_i^a we ensure that M_i^a does not require the same edge of G_{free} twice. Furthermore, when $a \in S$, by including all vw with $v \in U^a \cap Z(w)$ we ensure that M_i^a does not add both arcs of any 2-cycle to J: we cannot add $\overrightarrow{xx}'$, $\overleftarrow{xx}'$ with $x' = \phi_{w'}(a) \in U^a \cap Z(w)$ and $x = \phi_w(a) \in U^a \cap Z(w')$ as $xwx'w'$ would be an $M_i^a Z_i^a M_i^a Z_i^a$. The lemma follows. □

Next we note for all $i \in [3]$, $w \in W_i$ that $B_i^a(w) \subseteq U_{i'}$, where $i' = i - 1_{a \notin S^+}$. We record some simple consequences of this observation.

- $Z(w) \cap U_{i+1} = \emptyset$ for any $w \in W_i$.
- $Z(x) \cap W_{i-1} = N_J^-(x) \cap U_{i-1} = N_J^+(x) \cap U_{i+1} = \emptyset$ for any $x \in U_i$.
- If $w \in W_i$ then $Z(w) \cap U_i$ only contains $\phi_w(a)$ with $a \in S^+$, so has size $\leq |S^+| \leq 2\Delta$.
- If $x \in U_i$, $b \in S$ then $N_J^+(x) \cap U_i^b = Z(M_i^b(x)) \cap U_i^b$ has size $\leq 2\Delta$, as $M_i^b(x) \in W_i$.
- If $x \in U_i^b$ then $Z(x) \cap W_i$ only contains $M_i^a(x)$ with $a \in S^+$, so $|Z(x) \cap W_i| \leq 2\Delta$.
- If $x \in U_i^b$ then $N_J^-(x) \cap U_i = M_i^b(Z(x) \cap W_i)$ has size $\leq 2\Delta$.
- Each Z_i^a has maximum degree $\leq 2\Delta$.

By construction, B_i^a is a balanced bipartite graph. To justify the application of Lemma 2.7 in choosing M_i^a it remains to establish the following.

Lemma 4.8. *B_i^a is $p_+^{.15}$-super-regular.*

Proof. We first consider $\Gamma_i^a \subseteq U_{i'} \times W_i$ with $i' = i - 1_{a \notin S^+}$ defined by $N_{\Gamma_i^a}(w) = U_{i'} \cap G(\phi_w(a^-))$. For any $R \in \binom{U_{i'}}{\leq 2}$ we have $N_{\Gamma_i^a}(R) = \{w \in W_i : R \subseteq G(\phi_w(a^-))\} = M_i^{a^-}(N_G(R) \cap U_{i'})$, so $|N_{\Gamma_i^a}(R)| = |N_G(R) \cap U_{i'}| = ((1 \pm 1.1\xi)p)^{|R|}|W_i|$ whp by typicality and a Chernoff bound. Similarly, $N_{\Gamma_i^a}(R') = U_{i'} \cap \bigcap_{w \in R'} G(\phi_w(a^-))$ for $R' \in \binom{W_i}{\leq 2}$ whp has size $((1 \pm 1.1\xi)p)^{|R'|}|U_{i'}|$.

Now we will show that $\Gamma_i^a \setminus B_i^a$ has maximum degree $\leq 5p_+ n$. To see this, we first note that we have a contribution $\leq 4|V(F)| \leq 4p_+ n$ to any degree in $\Gamma_i^a \setminus B_i^a$ due to edges in Z or $G \setminus G_{\text{free}}$ (including the ≤ 1 vertex that is the image of $a \notin S^+$ for two w, w'). There are no other contributions for $a \notin S^+$, so we consider $a \in S^+$ and so $i' = i$. First we estimate the contribution to degrees of $w \in W_i$ and to degrees of $x \in U_i$ due to $x \in N_J^-(\phi_w(b) \cap U_i^b)$ for $b \in S$, which we claim are both $\leq 4\Delta^2$. Indeed, for $w \in W_i$ the contribution is $\leq \sum_{b \in S} |N_J^-(\phi_w(b)) \cap U_i| \leq 2|S|\Delta \leq 4\Delta^2$. For $x \in U_i$, we count $w \in W_i$ if $\phi_w(b) = y \in N_J^+(x)$, where $y \in U_i$ as $w \in W_i, b \in S$, so this contribution is $\leq \sum_b |N_J^+(x) \cap U_i^b| \leq 4\Delta^2$.

It remains to estimate the contribution to degrees of $w \in W_i$ and to degrees of $x \in U_i$ due to $x \in N_J^+(Z(w) \cap U^a)$, which we claim are both $\leq 8\Delta^3$. To see this, first note that we must have

$w \in Z(y)$ for some $y \in N_J^-(x) \cap U^a$, and $x \in Z(w') \cap U_i^b$ for some b with $\phi_{w'}(b) = y \in Z(w)$. We note that $w' \in W_i$, as otherwise $x \in Z(w')$ implies $w' \in W_{i+1}$ and $y = \phi_{w'}(b)$ implies $y \in U_{i+1}$, which contradicts $y \in Z(w)$. Thus we have $\leq 2\Delta$ choices for each of $w' \in Z(x) \cap W_i$, then $y \in Z(w') \cap U_i$, then $w \in Z(y) \cap W_i$, which proves the claim. The lemma now follows from Lemma 2.5. □

Thus we can apply Lemma 2.7, so each $M_i^a = \{\phi_w(a)w : w \in W_i\}$=MATCH$(B_i^a, Z_i^a)$ can be chosen and has $\mathbb{P}(vw \in M_i^a) = (1 \pm p_+^{.1})(pn)^{-1}$ for all $vw \in B_i^a$. In particular, we can complete step (iv), thus choosing edge-disjoint copies F_w of F.

Lemma 4.9. *For $x \in V$, $w \in W$, $a \in S$ whp $|U^a \cap V(F_w)| < 1.1p_+|U^a|$ and $|N_J^\pm(x) \cap U^a| < .1p_+^{.9}|U^a|$.*

Proof. The first statement holds by Lemma 2.4, as $|U^a \cap V(F_w)|$ is $(\mu, 1)$-dominated with

$$\mu = (1 \pm p_+^{.1})(pn)^{-1} \sum_{u \in V(F)} |B_i^u(w) \cap U^a| = (1 \pm 1.1p_+^{.1})|V(F)|n^{-1}|U^a|.$$

Next recall for $x \in U_i$ that $N_J^-(x) \cap U_{i-1} = N_J^+(x) \cap U_{i+1} = \emptyset$, $|N_J^-(x) \cap U_i| \leq 2\Delta$ and $|N_J^+(x) \cap U_i| = \sum_b |N_J^+(x) \cap U_i^b| \leq 4\Delta^2$.

To bound $|N_J^+(x) \cap U_{i-1}^a|$, note that for any $w \in W$ there are $< 1.1p_+|U^a|$ choices of $x' \in U^a \cap V(F_w)$, for which we add $\overrightarrow{xx'}$ to J if M_i^a chooses xw. Thus $|N_J^+(x) \cap U_{i-1}^a|$ is $(\mu, 1)$-dominated with $\mu < n \cdot 1.1p_+|U^a| \cdot (1 \pm p_+^{.1})(pn)^{-1}$, so by Lemma 2.4 whp $< .01p_+^{.9}|U^a|$.

Finally, for $x \in U_i^b$ we have $N_J^-(x) \cap U_{i+1}^a = M^b(Z(x)) \cap U_{i+1}^a$, which by Lemma 2.7 whp has size $|M_{i+1}^b[Z(x) \cap W_{i+1}, U_{i+1}^a]| < |Z(x)||U_{i+1}^a|/.99p|W_{i+1}| + n^{.8} < .01p_+^{.9}|U^a|$. □

We deduce $|N_J^\pm(x)| < .1p_+^{.9}n$, so the underlying graph $\widetilde{J}$ of J has maximum degree $< .2p_+^{.9}n$.

In step (v) we orient G_{free} as $D = \bigcup_{w \in W} D_w$, where for each $xy \in G_{\text{free}}$ with $x \in U^a$ and $y \in U^b$, if $\overrightarrow{xy} \in J$ we have $\overrightarrow{yx} \in D_w$ where $\phi_w(a) = y$, if $\overrightarrow{yx} \in J$ we have $\overrightarrow{xy} \in D_w$ where $\phi_w(b) = x$, or otherwise we make one of these choices independently with probability $1/2$. We define $Z^+ \subseteq V \times W$ by $Z^+(w) = V(F_w) \cup V(D_w)$.

Lemma 4.10. *whp $d_{D_w}^+(x)$ and $|Z^+(w) \cap U^a|$ are $(1 \pm p_+^{.8})d_{\mathcal{S}}(a)$ for all $x = \phi_w(a)$, $w \in W$, $a \in S$.*

Proof. First note by typicality and Chernoff bounds that whp there are $(1 + 2\xi)nd_{\mathcal{S}}(a)p/|\mathcal{S}| \pm 1.1p_+|U^a| = (1 \pm p_+^{.85})2d_{\mathcal{S}}(a)$ choices of $v \in U^a \cap G_{\text{free}}(x)$ after step (iv). Excluding $< .2p_+^{.9}|U^a|$ choices with $xv \in \widetilde{J}$, for all other v independently $\mathbb{P}(\overrightarrow{xv} \in D_w) = 1/2$. The lemma follows by a Chernoff bound and Lemma 4.9. □

To analyse step (vi), we first observe that initially the sets $N_{D_w}^+(\phi_w(a))$ are disjoint over $a \in S$ and disjoint from $V(F_w)$, and this is preserved by each move; moreover, each move decreases Σ by 2, and if (vi) does not abort we have every $D_w \cup F_w = T$. So it suffices to show that (vi) does not abort. We start with an estimate for the number of moves for any $uwu'w'$ that are *original*, meaning that they are present at the end of step (v) before any arcs are moved.

Lemma 4.11. *Any $u = \phi_w(a)$, $u' = \phi_{w'}(a')$ whp have $> 9000^{-1}p^3n^2d_{\mathcal{S}}(a')$ original $uwu'w'$-moves.*

Proof. We estimate the number of moves by sequentially choosing x, v then z. Suppose $u \in U^b$ and $w' \in W_i$, so $u' = \phi_{w'}(a') \in U_i$. Suppose $u' \in U^{b'}$. We claim there are whp $> .08pn$ choices of $x \in U_i \cap N_D^-(u) \setminus Z^+(w)$. To see this, note that there are $(1 \pm p_+^{.8})pn/3$ choices of $x \in U_i \cap G_{\text{free}}(u)$. Excluding $< p_+^{.9}n$ with xu or $x\phi_w(b)$ in $\widetilde{J}$, for all others independently $\mathbb{P}(x \in N_D^-(u) \setminus Z^+(w)) \geq 1/4$, so the claim holds by a Chernoff bound. Consider any such x, say with $x \in U_i^c$, and let $w^x = M^b(x) \in W_i$.

We claim there are whp $> .01p^2 n$ choices of $v \in U_{i-1} \cap N_D^-(x) \cap N_D^+(u') \setminus (Z^+(w') \cup Z^+(w^x) \cup M^c(Z^+(u')))$. To see this, note that there are $(1 \pm p_+^{.8})p^2 n/3$ choices of $v \in U_{i-1} \cap G_{\text{free}}(x) \cap G_{\text{free}}(u')$. For any such v, say in U_{i-1}^d, we have $v' := \phi_{w'}(d) \in U_i$, so $\overrightarrow{vv'} \notin J$, and so $\mathbb{P}(\overleftarrow{vv'} \in D) \leq 1/2$. Similarly, $v^x := \phi_{w^x}(d)$ has $\mathbb{P}(\overleftarrow{vv^x} \in D) \leq 1/2$ independently. Also $v \in M^c(Z^+(u') \setminus Z(u')) \Leftrightarrow w^v := M^c(v) \in Z^+(u') \setminus Z(u') \Leftrightarrow \overrightarrow{y_v u'} \in D$, where $y_v = \phi_{w^v}(b') \in U_{i-1}$ as $w^v \in W_{i-1}$. Since each v corresponds to a unique y_v, the number of y_v with $\overleftarrow{y_v u'} \in J$ is $d_J^+(u') \leq .1p_+^{.9}n$. Excluding such v, for all others we have $\overleftarrow{y_v u'} \notin J$ and so $\mathbb{P}(v \in M^c(Z^+(u') \setminus Z(u'))) \leq 1/2$ independently. Excluding a further $< p_+^{.9}n$ choices of v in $Z(w') \cup Z(w^x) \cup M^c(Z(u'))$ or with vx or vu' in $\widetilde{J}$, any other v contributes independently with probability $\geq 2^{-5}$, so the claim follows by a Chernoff bound.

Fix any such v, say with $v \in U^d$, and let $w^{u'} = M^d(u') \in W_{i-1}$. Similarly to the above, there are whp $> .16 d_{\mathcal{S}}(a')$ choices of $z \in U_{i-1} \cap N_{D_{w'}}^+(u') \setminus Z^+(w^{u'})$, as there are $(1 \pm p_+^{.8})d_{\mathcal{S}}(a')/3$ choices in $U_{i-1}^{a'} \cap G_{\text{free}}(u')$ and letting $z' = \phi_{w^{u'}}(a') \in U_i$, excluding at most $.1p_+^{.9}|U^{a'}|$ vertices z such that $\overleftarrow{zz'} \in J$, each other z has $\mathbb{P}(\overleftarrow{zz'} \in D) \leq 1/2$. The lemma follows. □

After t moves, we let $\mathcal{B}_t$ denote the bad event that any vertex y is incident to $> p_+^{.7}n$ moved arcs or to $> p_+^{.7}|U^q|$ arcs $\overrightarrow{yy'}$ with $y' \in U^q$ for some q. We let τ be the smallest t such that $\mathcal{B}_t$ occurs, or ∞ if there is no such t. At any step $t < \tau$ requiring a move for some $uwu'w'$ with $\phi_{w'}(a') = u'$, as $\mathcal{B}_t$ does not hold there are $> 10^{-4}p^3 n^2 d_{\mathcal{S}}(a')$ moves. To complete the proof it therefore suffices to show whp $\tau = \infty$. We fix t and bound $\mathbb{P}(\tau = t)$ as follows.

We start by showing that whp $< p_+^{.7}n$ arcs are moved at any y. To see this, note first that the number of times y plays the role of u or u' in a move is $\sum_{w \in W} |d_{D_w}^+(y) - d_{\mathcal{S}}(\phi_w^{-1}(y))| < \sum_a p_+^{.8} d_{\mathcal{S}}(a) < p_+^{.8}n$. Now fix $uwu'w'$ with $\phi_{w'}(a') = u'$. Then y plays the role of x or v in $< n d_{\mathcal{S}}(a')$ moves, so with probability $< 10^4 p^{-3} n^{-1}$. The number of moves where y plays x or v is therefore $(\mu, 1)$-dominated, where $\mu < p_+^{.8} n^2 \cdot 10^4 p^{-3} n^{-1} < .1p_+^{.7}n$, so by Lemma 2.4 whp $< .2p_+^{.7}n$. Furthermore, $y \in U^{a'}$ plays the role of z in $< n^2$ moves, so with probability $< 10^4 p^{-3} d_{\mathcal{S}}(a')^{-1}$. The number of such moves is therefore $(\mu, 1)$-dominated, where $\mu < 10^4 p^{-3} d_{\mathcal{S}}(a')^{-1} \sum_w |d_{D_w}^+(\phi_w(a')) - d_{\mathcal{S}}(a')| < .1p_+^{.7}n$, so by Lemma 2.4 whp $< .2p_+^{.7}n$. The claim follows.

Now, given $uwu'w'$, any arc $\overrightarrow{yy'}$ with $y' \in U^q$ plays the role of $\overleftarrow{vu'}$ or $\overrightarrow{xu}$ in $< n d_{\mathcal{S}}(q)$ moves, so with probability $< 10^4 p^{-3} n^{-1}$, and the role of $\overrightarrow{vx}$ in $< d_{\mathcal{S}}(q)$ moves, so with probability $< 10^4 p^{-3} n^{-2}$, and the role of $\overleftarrow{zu'}$ in $< n^2$ moves, so with probability $10^4 p^{-3} d_{\mathcal{S}}(q)^{-1}$. Thus for any $q \in S$ and $y = \phi_w(q)$, the number of moved $\overrightarrow{yy'}$ with $y' \in U^q$ is $(\mu, 1)$-dominated, where $\mu < 10^4 p^{-3} n^{-1} \cdot p_+^{.8} n |U^q| + 10^4 p^{-3} n^{-2} \cdot p_+^{.8} n^2 |U^q| + 10^4 p^{-3} d_{\mathcal{S}}(q)^{-1} \cdot |d_{D_w}^+(y) - d_{\mathcal{S}}(q)||U^q| < .1p_+^{.7}|U^q|$, so by Lemma 2.4 is whp $< p_+^{.7}|U^q|$. This completes the proof.

5 Concluding remarks

In this paper we have developed a variety of embedding techniques that are sufficiently flexible to resolve a generalised form of Ringel's Conjecture that applies to quasirandom graphs, and which promise to have more general applications to packings of a family of trees, as would be required for a solution of Gyárfás' Conjecture.

References

[1] A. Adamaszek, P. Allen, C. Grosu and J. Hladký, Almost all trees are almost graceful, arXiv:1608.01577 (2016).

[2] P. Allen, J. Böttcher, D. Clemens and A. Taraz, Perfectly packing graphs with bounded degeneracy and many leaves, arXiv:1906.11558 (2019).

[3] J. Barát, A. Gyárfás and G.N. Sárközy, Rainbow matchings in bipartite multigraphs, *Periodica Math. Hungar.* 74:108–111 (2017).

[4] S. Boucheron, G. Lugosi and P. Massart, *Concentration inequalities: a nonasymptotic theory of independence*, Oxford University Press (2016).

[5] P. Allen, J. Böttcher, J. Hladký and D. Piguet, Packing degenerate graphs, arXiv:1711.04869 (2017).

[6] J. Böttcher, J. Hladký, D. Piguet and A. Taraz, An approximate version of the tree packing conjecture, *Israel J. Math.* 211:391–446 (2016).

[7] B. Csaba, D. Kühn, A. Lo, D. Osthus and A. Treglown, Proof of the 1-factorization and Hamilton decomposition conjectures, *Mem. American Math. Soc.* 244, monograph 1154 (2016).

[8] S. Ehard, S. Glock and F. Joos, Pseudorandom hypergraph matchings. arXiv:1907.09946 (2019).

[9] A. Ferber, C. Lee and F. Mousset, Packing spanning graphs from separable families, *Israel J. Math.* 219:959–982 (2017).

[10] A. Ferber and W. Samotij, Packing trees of unbounded degrees in random graphs, *J. Lond. Math. Soc.* (2016).

[11] D. A. Freedman, On tail probabilities for martingales, *Ann. Probab.* 3:100–118 (1975).

[12] S. Glock, D. Kühn, A. Lo and D. Osthus, The existence of designs via iterative absorption, arXiv:1611.06827.

[13] S. Glock, D. Kühn, A. Lo and D. Osthus, Hypergraph F-designs for arbitrary F, arXiv:1706.01800.

[14] A. Gyárfás and J. Lehel, Packing trees of different order into K_n, in *Combinatorics* (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Volume 1, pages 463–469 (1978).

[15] F. Joos, J. Kim, D. Kühn and D. Osthus, Optimal packings of bounded degree trees, to appear in *J. European Math. Soc.*

[16] J. Kahn, A linear programming perspective on the Frankl-Rödl-Pippenger Theorem, *Random Struct. Alg.* 8:149–157 (1996).

[17] P. Keevash, The existence of designs, arXiv:1401.3665 (2014).

[18] P. Keevash, The existence of designs II, arXiv:1802.05900 (2018).

[19] P. Keevash, Coloured and directed designs, arXiv:1807.05770 (2018).

[20] P. Keevash and K. Staden, The generalised Oberwolfach problem, preprint (2020).

[21] J. Kim, D. Kühn, D. Osthus and M. Tyomkyn, A blow-up lemma for approximate decompositions, *Trans. Amer. Math. Soc.* 371:4655–4742 (2019).

[22] T. P. Kirkman, On a problem in combinations, *Cambridge and Dublin Math. J.* 2:191–204 (1847).

[23] Y. Kohayakawa and V. Rödl, Algorithmic aspects of regularity, Latin American Theoretical Informatics (LATIN 2000, Punta del Este), *Lecture Notes in Computer Science* 1776:1–17, Springer, Berlin, 2000.

[24] S. Messuti, V. Rödl and M. Schacht, Packing minor-closed families of graphs into complete graphs, *J. Combin. Theory B* 119:245–265 (2016).

[25] R. Montgomery, Spanning trees in random graphs, to appear in *Adv. Math.*

[26] R. Montgomery, A. Pokrovskiy and B. Sudakov, Decompositions into spanning rainbow structures, *Proc. London Math. Soc.* 119:899–959 (2019).

[27] R. Montgomery, A. Pokrovskiy and B. Sudakov, Embedding rainbow trees with applications to graph labelling and decomposition, to appear in *J. European Math. Soc.*

[28] R. Montgomery, A. Pokrovskiy and B. Sudakov, A proof of Ringel's conjecture, arXiv:2001.02665 (2020).

[29] D. K. Ray-Chaudhuri and R. M. Wilson, Solution of Kirkman's schoolgirl problem, *Combinatorics* (Proc. Sympos. Pure Math., Vol. XIX, Univ. California, Los Angeles, Calif., 1968), Amer. Math. Soc., Providence,R.I., 187–203 (1971).

[30] G. Ringel, Theory of graphs and its applications, in *Proc. Symp. Smolenice* (1963).

[31] V. Rödl, On a packing and covering problem, *Europ. J. Combin.* 6:69–78 (1985).

[32] R. M. Wilson, An existence theory for pairwise balanced designs I. Composition theorems and morphisms, *J.Combin. Theory A* 13:220–245 (1972).

[33] R. M. Wilson, An existence theory for pairwise balanced designs II. The structure of PBD-closed sets and the existence conjectures, *J. Combin. Theory A* 13:246–273 (1972).

[34] R. M. Wilson, An existence theory for pairwise balanced designs III. Proof of the existence conjectures, *J.Combin. Theory A* 18:71–79 (1975).

[35] R. M. Wilson, Decompositions of complete graphs into subgraphs isomorphic to a given graph, in *Proc. 5th British Combin. Conf.* (1975).